ABSTRACT

Title of Dissertation: DYNAMICS OF FERMIONIC MANY-BODY SYSTEMS

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In the thesis, we describe the dynamics of two many-body systems for Fermions.

The first system depicts infinitely many electrons moving in a constant magnetic field, where they interact with each other. In the reduced Hartree-Fock setting (ignoring the exchange term in the Hartree-Fock setting), the main part of electrons occupy the low energy state and are stationary, while the small part of them are excited particles. Based on this setting, we used Harmonic analysis and asymptotic properties of associated Laguerre polynomials to establish a local well-posedness theory, which is below the energy level.

The second system describes the motion of finitely many Fermions in the absence of background fields. In the Bogoliubov-de Gennes setting, based on the observation that the correlation function modeling Cooper pairs is anti-symmetric, we were able to employ techniques of dispersive equations and extend the existing global well-posedness theory. The well-posedness theory was proven for the Coulomb interaction potential. We extended it to the case with a more singular interaction

potential $\frac{1}{|x|^{2-\epsilon}}$, for any $0 \leq \epsilon < 2$.

# DYNAMICS OF FERMIONIC MANY-BODY SYSTEMS

by

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# Dedication

To my Mom and Dad

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## List of Symbols and Abbreviations

| | |
|---|---|
| $[A, B]$ | the commutator $AB - BA$ |
| $[A, B]_+$ | the anticommutator $AB + BA$ |
| $\|T\|_{op}$ | the operator norm of operator $T$. |
| $A \lesssim B$ | there is a constant $C$ such that $A \leq CB$. |
| $A \lesssim_{p,q} B$ | the constant $C$ depends on parameters $p$ and $q$. |
| $A \sim B$ | there are constants $C_1$ and $C_2$ such that $C_2 > C_1 > 0$ and $C_1 A \leq B \leq C_2 A$. |
| $A \sim_{p,q} B$ | constants $C_1$ and $C_2$ depend on $p$ and $q$. |
| $\mathcal{D}(H)$ | the domain of operator $H$. |
| $\mathcal{S}(\mathbb{R}^d)$ | the Schwartz space on $\mathbb{R}^d$. |
| $\langle\nabla\rangle^s$ | let $f \in \mathbb{R}^d$, $\left(\langle\nabla\rangle^s f\right)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{s/2} \hat{f}(\xi) e^{ix\xi} d\xi$, where $\hat{f}$ denotes the Fourier transform of $f$. |
| $\mathcal{L}^{s,p}$ | the Schatten-Sobolev norm of an operator $k$ $\|k\|_{\mathcal{L}^{s,p}} := \mathrm{Tr}\left(\|\langle\nabla\rangle^s k \langle\nabla\rangle^s\|^p\right)^{1/p}$ where the trace is taken over $L^2(\mathbb{R}^d)$. When $s = 0$, $\mathcal{L}^{0,p}$ is the usual Schatten norm and it is denoted as $\mathcal{L}^p$ for simplicity. |
| $\rho_k(x)$ | the diagonal $k(x, x)$ of $k(x, y)$, where $x, y \in \mathbb{R}^d$. |
| $\|k(t, x, y)\|_{L^p_t L^r_x L^l_y}$ | the norm $\|k(t, x, y)\|_{L^p_t(\mathbb{R}, L^r_x L^l_y)}$. Furthermore if the product domain is specified, $\|k(t, x, y)\|_{L^p_t L^r_x L^l_y(I \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})} := \|k(t, x, y)\|_{L^p_t(I, L^r_x L^l_y)}$, where $L^r_x$ and $L^l_y$ are short for $L^r_x(\mathbb{R}^{d_1})$ and $L^l_y(\mathbb{R}^{d_2})$ respectively. |
| HFB | Hartree-Fock-Bogoliubov |
| RHF | Reduced-Hartree-Fock |
| PDE | partial differential equation |
| CAR | canonical anticommutation relations |

# Chapter 1: Introduction

Nowadays, the Hartree-Fock approximation is a standard tool in quantum chemistry. It is a self-consistent field method to compute the approximation to the ground quantum state of a quantum many-body system. The main equations of the method are the Hartree-Fock equations and they are used to solve a time-independent problem. There are already numerous applications of this method. I am interested in the time-dependent problem. Consider the time-dependent Hartree-Fock equation [Dir30] in the density matrix formulation for a system consisting of interacting Fermions. The Hartree-Fock equation provides an approximation scheme to the many-body Schrödinger equation for quasi-free states.

The existence problem of the time-dependent Hartree-Fock equation has attracted a lot of attentions. In dimension three, when the one-particle Hamiltonian is the kinetic operator, i.e. it is the Laplace operator $-\Delta$, Bove-Prato-Fano [BDPF74] first showed the there is a unique mild solution to the Hartree-Fock equation if the two-body interaction potential is bounded. They later extended their result to the case [BDPF76][1] when the two-body interaction potential is dominated by the kinetic part $-\Delta$ by using the theory of semigroups. By the virtue of Hardy's

[1] In this paper, the one-particle Hamiltonian can include the Coulomb potential

inequality, their two-body interaction includes the Coulomb potential case. In the same year, Chadam [Cha76] independently obtained the global well-posedness result for Coulomb potential using a limiting argument. In 1992, Zagatti [Zag92] used Strichartz estimates and showed the global well-posedness of the Hartree-Fock equation when the one-particle Hamiltonian is $-\Delta+\mathrm{V}$, where $V$ and the two-body interaction potential are singular and they satisfy mixed type $L^p$ conditions. In dimension three, the two-body interaction potential in [Zag92] can be as singular as $1/|x|^{2-\epsilon}$ for arbitrarily small $\epsilon > 0$.

In the last two decades, there is a large literature of studying the effective dynamics of Bosonic many-body systems as the number of particles goes to infinity. The limiting behavior of effective dynamics is expected to capture the main properties of the many-body Schrödinger equation. We refer to Chong's thesis [Cho19] for detailed discussion. Similar work has also been done for Fermionic many-body systems. Several groups of authors established mean-field (with possible different scalings) approximation to the many-body Schrödinger equation using the Hartree-Fock equation. They compared one-particle density matrices for the two types of equations and showed the limit of the difference vanishes as the number of particles goes to infinity. More specifically, Bardos-Golse-Gottlieb-Mauser [BGGM03] for the case that the initial state is close to a Slater determinant and two-body interaction potential is bounded, Fröhlich-Knowles [FK11] for the case when the initial state is a Slater determinant and the two-body interaction potential is Coulomb, Benedikter-Porta-Schlein [BPS14] for the case when the initial state is close to a Slater determinant and the two-body interaction potential is sufficiently regular, and Benedikter-Jakšić-Porta-Saffirio-Schlein [BJP+16] for the case when the initial data is close to a quasi-free state and the two-body interaction potential is sufficiently regular. The dynamics of the many-body Schrödinger equation can also be effectively described by the Vlasov equation [Spo81, NS81].

In the thesis, we describe two variations of the Hartree-Fock equation: a reduced version for a system of infinitely many Fermions; a more complicated version for quasi-free states.

The reduced version of Hartree-Fock equation is the Hartree equation, which is derived by omitting exchange term from Hartree-Fock equation. As shown in [Sol91, EESY04, BPS14], in the mean-field limit, the exchange term of the Hartree-Fock equation is of low order. Removing exchange term, the resulting Hartree equation can still be used to describe the quantum system effectively. The Hartree equation demonstrates distinct properties from Hartree-Fock equation. Lewin-Sabin and Chen-Hong-Pavlović [LS15, LS14, CHP17, CHP18] proved there are stationary solutions to the Hartree equation and the equation is well-posed near the stationary solutions, where the stationary solutions are directly related to the Fermi-Dirac distribution. The density matrices of the stationary solutions are not in trace class (they are not even compact operators). Therefore the density matrix of the whole system is not of trace class and formally it corresponds to an infinite many-body system. Motivated by their work, we considered the Hartree equation for a model of infinitely many electrons in a constant magnetic field and "project" the equation of the system to dimension two. we proved that there are two families of stationary solutions and the Hartree equation is locally well-posed near one family of the stationary solutions, which is related to the Fermi-Dirac distribution. The presence of the constant magnetic field discretizes the spectrum[2] of the one-particle Hamiltonian: Pauli operator, and the Pauli operator does not dominate the Laplace operator as the Harmonic oscillator. We can not handle the problem as in the case when there is no background field. Furthermore, that the stationary solutions are not of trace class also causes issues when we apply dispersive PDE techniques. We introduced the Fourier-Wigner transform to our problem and used the asymptotic properties of associated Laguerre polynomials to derive a collapsing estimate, whose counterpart for the Laplace case was obtained by [GM17, CH16, CHP17, Cho18]. Using this ingredient, we obtained the local result.

The more complicated version of Hartree-Fock equation is the Bogoliubov-de Gennes equations, which describes the evolution of two-particle correlation functions: the one-particle density matrix and the pairing function (for Cooper pairs). Benedikter-Sok-Solovej [BSS18] formulated the Dirac–Frenkel approximation principle in terms of reduced density matrices and applied it to the Fermionic system. They obtained the Bogoliubov-de Gennes equations as an approximation to the many-body Schrödinger equation and the approximation is optimal within the class of pure quasifree states. Motivated by the work of Grillakis-Machedon [GM13, GM17], one can also derive the Bogoliubov-de Gennes equations for pure quasi-free states in a slightly different way. However using the Dirac–Frenkel principle, we can naturally generalize from pure quasi-free states to all quasi-free states. No matter which type of quasi-free states is taken into consideration: pure or mixed,

[2] The spectrum are Landau levels.

the Bogoliubov-de Gennes equations for two-particle correlation functions are of the same form. My work is to extend the existing global well-posedness result of Benedikter-Sok-Solovej from the Coulomb potential to $1/|x|^{2-\epsilon}$ for arbitrarily small $\epsilon > 0$. Intuitively, by the Pauli exclusion principle, the pairing function $\Lambda(x, y)$ vanishes on the diagonal $y = x$ and we should be able to deal with more singular two-body interaction potentials than the Coulomb potential. Mathematically, we used dispersive PDE techniques and the Morrey's inequality for Banach spaces and successfully handled $1/|x - y|^{2-\epsilon}\Lambda(x, y)$ in the Bogoliubov-de Gennes equations.

The thesis is organized as follows: we presented the Fock space formulation and main results of the two models in Chapter 2. In Chapter 3, we proved the well-posedness result for the Hartree equation with constant magnetic field. In Chapter 4, we established the global well-posedness result for the Bogoliubov-de Gennes equations. In the appendix, we discussed the Clifford algebra representation and the structure of pure quasi-free states.

# Chapter 2: Main Results

## 2.1 Fock Space Formulation

The setting of our problem is the Fermionic Fock space. Let $L_a^2(\mathbb{R}^{3n})$ denote the $L^2$-subspace of anti-symmetric functions. The Fermionic Fock space $\mathcal{F}$ is a Hilbert space consisting of vectors in the form

$$|\psi\rangle = \left(\psi^0, \psi^1, \psi^2, \dots\right),$$

where $\psi^0 \in \mathbb{C}$ and $\psi^j \in L^2(\mathbb{R}^{3j})$, $j \geq 1$. The inner product on $\mathcal{F}$ is defined as

$$\langle \varphi, \psi \rangle_{\mathcal{F}_a} := \sum_{j=0}^{\infty} \langle \varphi^j, \psi^j \rangle_{L_a^2(\mathbb{R}^{3n})}, \quad \varphi, \psi \in \mathcal{F}_a,$$

where $\langle \varphi^j, \psi^j \rangle_{L_a^2(\mathbb{R}^{3n})} = \int \bar{\varphi}^j \psi^j$ are inner products on $L_a^2(\mathbb{R}^{3n})$. In a word, the Fermionic Fock space $\mathcal{F}_a$ is the norm completion of the direct sum

$$\mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L_a^2\left(\mathbb{R}^{3n}\right)$$

with the given inner product. The vacuum state $(1, 0, 0, \dots)$ is denoted as $|0\rangle$. Every subspace $L_a^2(\mathbb{R}^{3n})$ of $\mathcal{F}_a$ is the state space for a system of $n$ Fermions. We can form antisymmetric $n$-particle functions by antisymmetrizing $n$ functions:

$$(f_1 \wedge \cdots \wedge f_n)(x_1, \ldots, x_n) := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} f_{\sigma(1)}(x_1) \cdots f_{\sigma(n)}(x_n) \tag{2.1}$$

where $S_n$ is the symmetric group of $\{1, 2, \ldots, n\}$, $\mathrm{sgn}(\sigma)$ denotes the sign of $\sigma$ and $f_j \in L^2(\mathbb{R}^3)$. We use physics bra–ket notations to denote operators, for example let $|\psi\rangle, |\varphi\rangle \in \mathcal{F}_a$, $|\psi\rangle\langle\varphi|$ acts on $\mathcal{F}_a$ as

$$|\psi\rangle\langle\varphi|(|\phi\rangle) := |\psi\rangle\langle\varphi, \phi\rangle_{\mathcal{F}_a}, \quad |\phi\rangle \in \mathcal{F}_a. \tag{2.2}$$

In this Fock space $\mathcal{F}_a$, we introduce creation and annihilation distribution valued operators and denote them by $a_x^\dagger$ and $a_x$ respectively. $a_x^\dagger$ and $a_x$ act on $L_a^2(\mathbb{R}^{3(n-1)})$ and $L_a^2(\mathbb{R}^{3(n+1)})$ in the following way,

$$\begin{aligned} a_x^\dagger(\psi^{n-1})(x_1, \ldots, x_n) &:= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j+1} \delta(x - x_j) \psi^{n-1}(x_1, \ldots, \hat{x}_j, \ldots, x_n), \\ a_x(\psi^{n+1})(x_1, \ldots, x_n) &:= \sqrt{n+1}\, \psi^{n+1}([x], x_1, \ldots, x_n), \end{aligned}$$

where $\psi^{n-1} \in L_a^2(\mathbb{R}^{3(n-1)})$, $\psi^{n+1} \in L_a^2(\mathbb{R}^{3(n+1)})$, $\hat{x}_j$ means the variable $x_j$ is ignored and $[x]$ indicates the variable $x$ is frozen. In addition, $a_x(\psi^0) = 0$ for $\psi^0 \in \mathbb{C}$. The creation and annihilation operators satisfy the canonical anticommutation relations (CAR)

$$[a_x, a_y]_+ = 0, \quad [a_x^\dagger, a_y^\dagger]_+ = 0 \quad \text{and} \quad [a_x, a_y^\dagger]_+ = \delta(x - y). \tag{2.3}$$

Using the distribution valued operators, we can form operators which act on the

Fock space $\mathcal{F}_a$ by introducing a field $\phi \in L^2(\mathbb{R}^3)$

$$a^\dagger(\phi) := \int \phi(x) a_x^\dagger \, dx \quad a(\phi) := \int \phi(x) a_x \, dx,$$

and for vectors of $L_a^2(\mathbb{R}^{3(n-1)})$ and $L_a^2(\mathbb{R}^{3(n+1)})$

$$a^\dagger(\phi)(\psi^{n-1})(x_1, \dots, x_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j+1} \phi(x_j) \psi^{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n),$$
$$a(\phi)(\psi^{n+1})(x_1, \dots, x_n) := \sqrt{n+1} \int_{\mathbb{R}^3} \phi(x) \psi^{n+1}(x, x_1, \dots, x_n) \, dx,$$

where $\psi^{n-1} \in L_a^2(\mathbb{R}^{3(n-1)})$ and $\psi^{n+1} \in L_a^2(\mathbb{R}^{3(n+1)})$. Note that $a(\phi)$ is complex linear in the parameter $\phi$.

## 2.2 Hartree Equation With Constant Magnetic Field: Well-Posedness Theory

In this section we present the first model: a system of infinitely many electrons moving in a constant magnetic field.

Without loss of generality, suppose the constant magnetic field $\mathbf{B} = (0, 0, b)$, $(b > 0)$. Let $\tilde{\mathfrak{h}} = (\sigma \cdot (-i\nabla - A))^2$ be the Pauli operator, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $A = -\frac{b}{2}\left(x^2, -x^1, 0\right)$ [1] is the vector potential of the field $\mathbf{B} = \nabla \times A$. The many-body Hamiltonian for a system of $N$ electrons moving in the constant

[1] There are other choices of $A$, for example $A = -b(x^2, 0, 0)$ [LL77, Chapter XV]. We use the one which is fixed by the Coulomb gauge $\nabla \cdot A = 0$.

magnetic field $\mathbf{B}$ is described by

$$\hat{H}_N = \sum_{j=1}^{N} \tilde{\mathfrak{h}}_j + \sum_{j>k} V\left(x_j - x_k\right), \quad x_j \in \mathbb{R}^3, \tag{2.4}$$

and the Schrödinger equation is

$$\begin{cases} i\,\partial_t \Psi_N(t, x_1, x_2, \ldots, x_N) = \hat{H}_N \Psi_N(t, x_1, x_2, \ldots, x_N) \\ \Psi_N(t=0) = \Psi_{N,0} \end{cases} \tag{2.5}$$

where $\Psi_{N,0} \in \wedge^N L^2\left(\mathbb{R}^3, \mathbb{C}^2\right)$: the space of antisymmetric functions, $\tilde{\mathfrak{h}}_j$ means $\tilde{\mathfrak{h}}$ acts on the variable $x_j$ (the $j$-th electron) and $V$ is the two-body interaction potential. A direct computation shows

$$\tilde{\mathfrak{h}} = \begin{pmatrix} (-i\nabla - A)^2 & 0 \\ 0 & (-i\nabla - A)^2 \end{pmatrix} - \sigma \cdot \mathbf{B},$$

while $\sigma \cdot \mathbf{B} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$ is harmless for the analysis of the system. For simplicity, we consider the scalar case, i.e.

$$\mathfrak{h} = (-i\nabla - A)^2. \tag{2.6}$$

If the initial data $\Psi_{N,0}$ is set to be a Slater determinant

$$\Psi_{N,0}(x_1, x_2, \ldots, x_N) = \psi_{1,0} \wedge \psi_{2,0} \wedge \cdots \wedge \psi_{N,0}(x_1, \ldots, x_N),$$

the corresponding Hatree-Fock equation in the density matrix formulation is

$$\begin{cases} i\,\partial_t \Gamma(t) = \left[\mathfrak{h} + \rho_{\Gamma(t)} * V - (V\Gamma)(t), \Gamma(t)\right], \\ \Gamma(0,x,y) = \Gamma_0(x,y) \end{cases} \tag{2.7}$$

where $\rho_{\Gamma(t)}(x) = \Gamma(t,x,x)$, $\rho_{\Gamma(t)} * V$ denotes the usual convolution, $(V\Gamma)(t,x,y) = V(x-y)\Gamma(t,x,y)$ is the exchange term and

$$\begin{aligned} \Gamma_0(x,y) :&= \left\langle \Psi_{N,0}, a_y a_x^\dagger \Psi_{N,0} \right\rangle_{L_a^2(\mathbb{R}^{3N})} \\ &= \sum_{j=1}^{N} \psi_{j,0}(x)\bar{\psi}_{j,0}(y), \quad x, y \in \mathbb{R}^3. \end{aligned}$$

After the time evolution, $\Psi_N(t)$ may not necessarily stay as a Slater determinant. Instead, one might expect that in an appropriate sense,

$$\Psi_N(t, x_1, \dots, x_N) \approx (\psi_1(t) \wedge \psi_2(t) \wedge \cdots \wedge \psi_N(t))\,(t, x_1, \dots, x_N),$$

However the density matrix $\Gamma(t)$ is still a projection and it is in the form

$$\Gamma(t,x,y) = \sum_{j=1}^{N} \psi_j(t,x)\bar{\psi}_j(t,y), \quad x, y \in \mathbb{R}^3, \tag{2.8}$$

where $\{\psi_j(t,x)\}_{j=1}^N$ remains an orthonormal set.

In a mean field regime and in the absence of the background magnetic field with a scaling of the kinetic part and the interaction part, Equation (2.7) is an effective description of Equation (2.5) for certain $V$ and initial data, when $N$ is sufficiently large. See details in [BPS14]. In [BPS14], the exchange term $(V\Gamma)(t)$ is of lower order and they also proved that the effective description remains true if Equation (2.7) is replaced by the following Hartree equation [2] in the reduced Hartree-Fock [Sol91] model. We omit the exchange term and obtain the Hartree equation

$$\begin{cases} i\,\partial_t \Gamma(t) = \left[\mathfrak{h} + \rho_{\Gamma(t)} * V, \Gamma(t)\right], \\ \Gamma(t=0) = \Gamma_0, \end{cases} \tag{2.9}$$

We refer to [BGGM03, EESY04, FK11] for other comparisons on the three dynamics from a perspective of mean field and semi-classical limit and refer to [NS81, Spo81] for a different mean field limit of Equation (2.5) on the Vlasov hierarchy.

The problem of our interest is the well-posedness theory of Hartree equations (2.9) when we take the formal limit of the number $N$ of particles to be infinite. Note that $\Gamma_0$ is not of trace class any more, but it still satisfies the operator inequality $0 \le \Gamma_0 \le 1$ which is due to the Pauli exclusion principle.

In the absence of magnetic fields, if $\Gamma_0$ is not of trace class, Equation (2.9) was recently studied by several authors [LS15, LS14, CHP17, CHP18] and they showed global well-posedness and the long time scattering behavior separately for different interaction potentials $V$.

In the presence of a constant magnetic field, to my knowledge, the author is the first one to consider the Hartree equation when $\Gamma_0$ is not of trace class or a

[2] They are called Hartree equations since the operator $\mathfrak{h} + \rho_{\Gamma(t)} * V$ is derived by applying the variational principle to the Hartree product $\psi_1 \otimes \cdots \otimes \psi_N$ instead of the Slater determinant $\psi_1 \wedge \cdots \wedge \psi_N$ [SO96, Chapter Three].

Hilbert-Schmidt operator. Since the operator $\mathfrak{h}$ is now the Pauli operator, which is different from the Laplace operator, the spectrum changes from a continuous one to a discrete one. Besides the eigenspaces of  are of infinite dimension. Even though we mainly care about the case when $\Gamma_0$ is not of trace class, to complete the picture, when $\Gamma_0$ is of trace class and $V(x) = \frac{1}{|x|}$, we establish a global well-posedness result at the energy level in the appendix 3.6.

The explicit form of Equation (2.9) is

$$i\,\partial_t \Gamma(t) = \left[-\partial_{x^3}^2 + D^* D + b + \rho_{\Gamma(t)} * V, \Gamma(t)\right], \tag{2.10}$$

where

$$D = -2\partial_{\bar{z}} - \frac{b}{2}z, \quad D^* = 2\partial_z - \frac{b}{2}\bar{z}, \quad z = x^1 + ix^2. \tag{2.11}$$

Consider first the two dimensional problem

$$\begin{cases} i\,\partial_t \gamma(t) = \left[H + \rho_{\gamma(t)} * v, \gamma(t)\right], \\ \gamma(0,x,y) = \gamma_0(x,y), \end{cases} \quad x, y \in \mathbb{R}^2, \tag{2.12}$$

where

$$H = D^* D, \quad \rho_\gamma(t,x) = \gamma(t,x,x), \tag{2.13}$$

and $\gamma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. If $v \in L^1(\mathbb{R}^2)$ [3], Equation (2.12) admits one family [4] of

[3]For the given family of solutions $\bar{\Pi}_\phi$, $\rho_{\bar{\Pi}_\phi} = \bar{\Pi}_\phi(x,x) = \phi(0)$ is constant. In order for $\rho_{\bar{\Pi}_\phi} * v$ to make sense, $v \in L^1(\mathbb{R}^2)$.

[4]For the other family, see Section 3.6.2.

non-trace class stationary solutions with integral kernels in the following form

$$\bar{\Pi}_{\phi}(x,y) = \phi\,(x-y)\,e^{-i\frac{b\Omega(x,y)}{2}}, \tag{2.14}$$

where $\Omega(x,y) := x^1y^2 - x^2y^1,\ x,y \in \mathbb{R}^2$ and $\phi$ is a radial symmetric function: $\phi(x) = \phi(|x|)$. The derivation is in Section 3.6.2.

Inspired by [LS15, LS14, CHP17, CHP18], we are interested in the evolution of perturbations of the stationary solutions. Suppose the pertubation of the stationary solution $\bar{\Pi}_{\phi}$ is $Q(t,x,y) = \gamma(t,x,y) - \bar{\Pi}_{\phi}(x,y)$, then we have the evolution equation for $Q$

$$\begin{cases} i\,\partial_t Q(t) = [H + \rho_{Q(t)} * v, Q(t)] + [\rho_{Q(t)} * v, \bar{\Pi}_{\phi}] \\ Q(0,x,y) = Q_0(x,y), \end{cases} \tag{2.15}$$

where $\rho_Q(t,x) = Q(t,x,x)$ and $x,y \in \mathbb{R}^2$.

Before we discuss the main results for Equation (2.15), we summarize the spectral property of the operator $H$ and explain the connection between the stationary solutions $\bar{\Pi}_{\phi}$ and the Fermi-Dirac distribution. The operator $H$ has a discrete spectrum $\sigma(H) = \{2bj\}_{j\in\mathbb{N}}$ on $L^2(\mathbb{R}^2)$ and its spectral decomposition is as follows

$$H = \sum_{j=0}^{\infty} 2bj\,P_j \tag{2.16}$$

where $P_j$ are mutually orthogonal projections onto eigenspaces corresponding to eigenvalues $2bj$. The eigenspace for each $2bj$ is of infinite dimension. More precisely,

$P_j$ have integral kernels

$$\frac{b}{2\pi} L_j \left( \frac{b}{2} |x - y|^2 \right) \exp\left( -\frac{b}{4} |x - y|^2 \right) e^{-i \frac{b\Omega(x,y)}{2}} \tag{2.17}$$

where $x, y \in \mathbb{R}^2$ and $L_k(\lambda)$ are Laguerre polynomials, i.e.

$$L_k(\lambda) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-\lambda)^j}{j!}, \ (\lambda \in \mathbb{R}). \tag{2.18}$$

For more details, see Section 3.2. From a functional calculus perspective, for the stationary solutions (2.14), $\phi$ corresponds to a function $l$ defined on the spectrum $\sigma(H)$. Let $l_j$ denote $l(2bj)$ and $l(H) \coloneqq \sum_{j=0}^{\infty} l\,(2bj)\, P_j$. Then $l$ corresponds to $\phi$ in (2.14):

$$\phi(x) = \frac{b}{2\pi} \sum_{j=0}^{\infty} l_j L_j \left( \frac{b}{2} |x|^2 \right) \exp\left( -\frac{b}{4} |x|^2 \right). \tag{2.19}$$

The Fermi-Dirac distributions at different temperatures provide important examples for the stationary solutions $\bar{\Pi}_\phi$. Let $k_B$ be the Boltzmann's constant and $T$ be the absolute temperature, the Fermi-Dirac distribution in the operator form is given by

$$\frac{1}{e^{(H-\mu)/k_B T} + 1} f \coloneqq \sum_{j=0}^{\infty} \frac{1}{e^{(2bj-\mu)/k_B T} + 1} P_j f, \tag{2.20}$$

where $f \in L^2(\mathbb{R}^2)$. When we set $\mu = 2nb$, the zero temperature limit $(T \to 0^+)$ of (2.20) is $\mathbf{1}_{(H \le 2nb)}$, which is exactly the projection $\bar{\Pi}_\phi$ with

$$\phi\,(x) = \frac{b}{2\pi} \sum_{j=0}^{n} L_j \left( \frac{b}{2} |x|^2 \right) \exp\left( -\frac{b}{4} |x|^2 \right), \quad x \in \mathbb{R}^2. \tag{2.21}$$

Now $\bar{\Pi}_\phi$ is the projection onto the first $n+1$ eigenspaces [5] of $H$, i.e. the possible lowest $n+1$ energy levels of $H$. As an analog of the classical picture of a Fermi sea, we call $\bar{\Pi}_\phi$ the Fermi sea. In the general case, for any finite non-negative $\mu$, the Fermi-Dirac distribution corresponds to $\bar{\Pi}_\phi$ with

$$\phi(x) = \frac{b}{2\pi}\sum_{j=0}^{\infty}\frac{1}{e^{(2bj-\mu)/k_BT}+1}L_j\left(\frac{b}{2}|x|^2\right)\exp\left(-\frac{b}{4}|x|^2\right), \quad x \in \mathbb{R}^2. \tag{2.22}$$

When it comes to which norm to use in our analysis, it is crucial to define quantities based on the Hamiltonian $H$. Because the stationary solution is not of trace class or Hilbert-Schmidt, it does behave well when we measure our data using other criteria. As above discussion, the Hamiltonian $H$ has a clear spectral structure and it is natural to define norms based on the spectral decomposition.

**Definition 2.1.** Suppose $f \in L^2(\mathbb{R}^2)$, $s \geq 0$,

$$\left\|H^{s/2}f\right\|_{L^2}^2 := \sum_{j=0}^{\infty}(2bj)^s\left\|P_jf\right\|_{L^2}^2, \quad \left\|\langle H\rangle^{s/2}f\right\|_{L^2}^2 := \sum_{j=0}^{\infty}\langle 2bj\rangle^s\left\|P_jf\right\|_{L^2}^2,$$
$$\left\|\bar{H}^{s/2}f\right\|_{L^2} := \left\|H^{s/2}\bar{f}\right\|_{L^2}, \quad \left\|\langle\bar{H}\rangle^{s/2}f\right\|_{L^2} := \left\|\langle H\rangle^{s/2}\bar{f}\right\|_{L^2},$$

where $\langle 2bj\rangle = \left(1+(2bj)^2\right)^{1/2}$ and $\bar{H}$ is the complex conjugation of $H$, i.e.

$$\bar{H} = \overline{D^*}\bar{D} = \left(2\partial_{\bar{z}} - \frac{b}{2}z\right)\left(-2\partial_z - \frac{b}{2}\bar{z}\right).$$

$H$ and its complex conjugation $\bar{H}$ have similar spectral structures but in an

[5] In the physics literature, they are called Landau levels.

"orthogonal" sense. Please see Figure 3.1 for details.

With respect to the new norms, we obtain a local well-posedness result of Equation (2.15). To state the result, recall that a mild solution of Equation (2.15) is a solution satisfying the integral equation

$$Q(t,x,y) = e^{-it(H_x - \bar{H}_y)} Q_0(x,y) - i \int_0^t e^{-it(H_x - \bar{H}_y)(t-\tau)} \left[\rho_Q * v, Q + \bar{\Pi}_\phi\right] d\tau. \quad (2.23)$$

The solution space for the Banach fixed point argument is $\mathbf{N}_T^H$ endowed with the norm,

**Definition 2.2.** Let $k(t,x,y)$ be a function $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, the norm $\mathbf{N}_T^H$ is defined as

$$\begin{aligned} \|k(t,x,y)\|_{\mathbf{N}_T^H} :=& \sup_{(q,r)\in Ad} \|\langle H_x\rangle^{1/2}\langle \bar{H}_y\rangle^{1/2} k(t,x,y)\|_{L_t^q L_x^r L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \\ &+ \sup_{(q,r)\in Ad} \|\langle H_x\rangle^{1/2}\langle \bar{H}_y\rangle^{1/2} k(t,x,y)\|_{L_t^q L_y^r L_x^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \\ &+ \|\langle \nabla_x\rangle^{9/8} \rho_{k(t)}(x)\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^2)}, \end{aligned}$$

where $T \in \mathbb{R}$ and

$$Ad = \left\{(q,r) \middle| \left(\frac{1}{q}, \frac{1}{r}\right) \text{ is in the line segment connecting } \left(\frac{1}{\infty}, \frac{1}{2}\right) \text{ and } \left(\frac{1}{4}, \frac{1}{4}\right)\right\}. \quad (2.24)$$

The first part of $\mathbf{N}_T^H$ is the Strichartz norm and the set (2.24) is a subset of admissible pairs $(q,r)$ which satisfy

$$\frac{1}{q}+\frac{1}{r}=\frac{1}{2},\quad 2<q\leq\infty. \tag{2.25}$$

The second part of $\mathbf{N}_T^H$ involves the collapsing term $\rho_Q$, whose estimate is the main new ingredient in this project. The theorem that we want to prove is as follows

**Theorem 2.3.** *Consider Equation (2.15) and suppose that $v\in L^1(\mathbb{R}^2)$ and*

$$\phi(x)=\phi(|x|),\quad \left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}<\infty,\quad x\in\mathbb{R}^2. \tag{2.26}$$

*If the initial data $Q_0(x,y)$ satisfies*

$$\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q_0(x,y)\right\|_{L^2_{x,y}}<\infty,$$

*then for sufficiently short time $T$, Equation (2.15) has a mild solution in the Banach space $\mathbf{N}_T^H$.*

*Remark* 2.4. $\left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}$ is essentially $\left\|D\bar{D}\phi\right\|_{L^2}+\|D\phi\|_{L^2}+\left\|\bar{D}\phi\right\|_{L^2}+\|\phi\|_{L^2}$. By the relation (2.19), the condition passes to $\{l_j\}$ as $\sum_{j=0}^{\infty}j^2l_j^2<\infty$. Thus (2.22) satisfies the condition (2.26).

*Remark* 2.5. For the Banach space $\mathbf{N}_T^H$, we can increase the size of the set $Ad$ as long as it does not include to endpoint $\left(\frac{1}{2},\frac{1}{\infty}\right)$. Consequently, the existence time may decrease.

Since the norm $\mathbf{N}_T^H$ contains $\|\langle\nabla_x\rangle^{9/8}\rho_{k(t)}(x)\|_{L^2_tL^2_x([0,T]\times\mathbb{R}^2)}$, the proof of Theorem 2.3 is based on the following collapsing estimate.

**Theorem 2.6** (Collapsing Estimate). *Suppose $\gamma(t,x,y) = e^{-i(H_x-\bar{H}_y)t}\gamma_0(x,y)$ is the solution to the linear equation*

$$\begin{cases} i\,\partial_t\gamma(t) = [H,\gamma(t)] \\ \gamma(0,x,y) = \gamma_0(x,y) \in L^2(\mathbb{R}^4), \end{cases} \tag{2.27}$$

*where $x, y \in \mathbb{R}^2$, the collapsing term $\rho_\gamma(t,x) = \gamma(t,x,x)$ satisfies*

$$\left\|\rho_{\gamma(t)}(x)\right\|_{L_t^2L_x^2([0,\pi/b]\times\mathbb{R}^2)} \lesssim_b \left\|\langle H_x\rangle^{s/2}\langle H_y\rangle^{s/2}\gamma_0(x,y)\right\|_{L^2(\mathbb{R}^4)}, \quad s > \frac{1}{2}, \tag{2.28}$$

*and*

$$\left\||\nabla_x|^c\rho_{\gamma(t)}(x)\right\|_{L_t^2L_x^2([0,\pi/b]\times\mathbb{R}^2)} \lesssim_{c,b} \left\|\langle H_x\rangle^{1/2}\langle \bar{H}_y\rangle^{1/2}\gamma_0(x,y)\right\|_{L^2(\mathbb{R}^4)}, \quad 0 \le c < \frac{5}{4}. \tag{2.29}$$

*Remark* 2.7. The estimate (2.29) is only stated for the time interval $[0,\pi/b]$. However, since the solution $\gamma(t,x,y)$ has a period $\pi/b$, by a patching argument, (2.29) holds for arbitrary large time interval $[-T,T]$, while the constant will depend on $T$.

This type of estimates has been established in [GM17, CH16, CHP17] for the Laplacian case, i.e. $i\,\partial_t\gamma(t) = [-\Delta,\gamma(t)]$. However the technique used in those papers does not apply to the current case. That method, in the spirit of [KM08], is to study the characteristic hypersurface, which is derived by applying the space-time Fourier transform after we collapse the solution $e^{it(\Delta_x-\Delta_y)}\gamma_0$ to the diagonal $y = x$. In our case, the time Fourier transform is replaced by the Fourier series. The new ingredients are the Fourier-Wigner transform and a refined estimate about the asymptotic property of associated Laguerre polynomials.

## 2.3 Global Well-Posedness for Bogoliubov-de Gennes Equations

In this section, we present the second model: a system of weakly interacting Fermions, where the expected number of particles is finite and there is no background field.

In the density matrix formulation, a pure state is an operator on the Fock space $\mathcal{F}_a$ which is in the form $|\psi\rangle\langle\psi|$, where $\psi \in \mathcal{F}_a$ and $\|\psi\|_{\mathcal{F}_a} = 1$. In general, a mixed state $\omega$ is a positive self-adjoint trace class operator on $\mathcal{F}_a$ with trace norm 1. The mixed state is in the form

$$\omega = \sum_{j=1}^{\infty} \lambda_j |\psi_j\rangle\langle\psi_j| \tag{2.30}$$

where $\lambda_j \geq 0$, $\sum_j \lambda_j = 1$ and $\psi_j \in \mathcal{F}_a$ are orthonormal. It can be understood in the sense, the probability distribution of the system is given by $(\psi_j, \lambda_j)$, where $\lambda_j$ is the probability that the system is in state $\psi_j$. The many-body Schrödinger equation of the system in the density matrix formulation is

$$i\,\partial_t \omega(t) = [\hat{H}, \omega(t)] \tag{2.31}$$

where the many-body Hamiltonian is

$$\hat{H} := \int -\Delta_x \delta(x-y) a_x^\dagger a_y \, dxdy + \frac{1}{2} \int v(x-y) a_x^\dagger a_y^\dagger a_y a_x \, dxdy,$$

and $v$ is the two-body interaction potential: $v(x) = v(-x)$ and $v(x) \in \mathbb{R}$. For short

$$\mathcal{V} := \frac{1}{2} \int v(x-y) a_x^\dagger a_y^\dagger a_y a_x \, dxdy$$

and

$$-\widehat{\Delta} := \int -\Delta_x \delta(x-y) a_x^\dagger a_y \, dxdy.$$

To see how the Fock space Hamiltonian $\hat{H}$ acts on $\mathcal{F}_a$, let $\psi^n \in L_a^2(\mathbb{R}^{3n})$,

$$\left(\hat{H}\psi^n\right)(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} (-\Delta_{x_j}) + \sum_{j<k} v(x_j - x_k)\right) \psi^n(x_1, \ldots, x_n).$$

Next we "project" the many-body Hamiltonian action onto the subspace of mixed states: quasi-free states, which will be defined shortly. Recall that two-particle correlation functions of a state $\omega$ are defined in the sense of distribution

$$\Gamma(x,y) := \mathrm{Tr}_{\mathcal{F}_a}\left(a_y^\dagger a_x \omega\right) = \sum_{j=1}^{\infty} \lambda_j \left\langle \psi_j, a_y^\dagger a_x \psi_j \right\rangle_{\mathcal{F}_a} \tag{2.32}$$

$$\Lambda(x,y) := \mathrm{Tr}_{\mathcal{F}_a}(a_y a_x \omega) = \sum_{j=1}^{\infty} \lambda_j \left\langle \psi_j, a_y a_x \psi_j \right\rangle_{\mathcal{F}_a} \tag{2.33}$$

where the trace $\mathrm{Tr}_{\mathcal{F}_a}$ is taken over $\mathcal{F}_a$. $\Gamma$ and $\Lambda$ are also called the one particle density matrix and pairing function respectively, where $\Lambda$ is used to model the Cooper pairs.

Other correlation functions can be defined similarly.

A mixed state $\omega$ is quasi-free if it satisfies the Wick's theorem, i.e. any of its correlation functions can be determined by the two-particle correlation functions in the way

$$\mathrm{Tr}_{\mathcal{F}_a}\left(a_1^{\#}a_2^{\#}\cdots a_{2n+1}^{\#}\omega\right) = 0 \tag{2.34}$$

$$\mathrm{Tr}_{\mathcal{F}_a}\left(a_1^{\#}a_2^{\#}\cdots a_{2n}^{\#}\omega\right) = \sum_{\sigma\in S_{ad}} \mathrm{sgn}(\sigma)\mathrm{Tr}_{\mathcal{F}_a}\left(a_{\sigma(1)}^{\#}a_{\sigma(2)}^{\#}\omega\right)\cdots\mathrm{Tr}_{\mathcal{F}_a}\left(a_{\sigma(2n-1)}^{\#}a_{\sigma(2n)}^{\#}\omega\right) \tag{2.35}$$

where $a_j^{\#}$ denotes an operator without specifying whether it is a creation or annihilation operator, $\mathrm{sgn}(\sigma)$ denotes the sign of permutation $\sigma$ and $S_{ad}$ is a subset of the symmetric group $S_{2n}$ such that

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1),\ \ \sigma(2k-1) < \sigma(2k).$$

To further explain the correspondence between quasi-free states $\omega$ and their two-particle correlation functions, it is more convenient to work on the generalized one-particle density matrices $S_\omega$, which are defined as operators on $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ such that

$$\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, S_\omega \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} = \mathrm{Tr}_{\mathcal{F}_a}\left(\left(a^\dagger(f_2)+a(g_2)\right)\left(a^\dagger(f_1)+a(g_1)\right)^*\omega\right) \tag{2.36}$$

where $*$ denotes the adjoint of an operator. More specifically,

$$S_\omega = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & 1 - \bar{\Gamma} \end{pmatrix},$$

where the notation $\bar{\Gamma}$ means the complex conjugation of the operator $\Gamma$, which is defined as

**Definition 2.8.** Let $T$ be an operator on $L^2(\mathbb{R}^3)$, the complex conjugation of $\bar{T}$ is

$$\bar{T}f := \overline{T\bar{f}}, \quad f \in L^2\left(\mathbb{R}^3\right).$$

If $T$ has an integral kernel $k(x,y)$, the kernel of $\bar{T}$ is $\bar{k}(x,y)$.

Using the definition (2.36) and CAR, one can show for any state $\omega$, the generalized one-particle density matrix $S_\omega$ satisfies [6]

$$S_\omega + \mathcal{J} S_\omega \mathcal{J} = id_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} \quad \text{and} \quad 1 \geq S_\omega^* = S_\omega \geq 0 \tag{2.37}$$

where $\mathcal{J}$ is a complex conjugation defined on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$

$$\mathcal{J}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}, \quad f, g \in L^2\left(\mathbb{R}^3\right).$$

[6]More generally, if one considers the $C^*$-algebra generated by $a^\dagger(f)$ and $a(g)$ for any $f, g \in L^2\left(\mathbb{R}^3\right)$ and states as positive normalized linear functionals over the $C^*$-algebra, where the normalization is that $\omega(e) = 1$ and $e$ is the identity element in the $C^*$-algebra, this result still holds. We refer interested readers to [Ara71], while our $S_\omega$ and $\mathcal{J}$ are the $1 - S$ and $\Gamma$ in [Ara71]. Note that all mixed states in our sense yield positive functionals.

**Definition 2.9.** Let $\mathcal{S}_{ad}$ denote

$$\mathcal{S}_{ad} := \left\{ S \middle| S + \mathcal{J} S \mathcal{J} = id_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} \quad \text{and} \quad 1 \geq S^* = S \geq 0 \right\} \tag{2.38}$$

For any $S \in \mathcal{S}_{ad}$, we can always associate $S$ with a quasi-free by [Ara71, Lemma 4.6]. The correspondence from the space of quasi-free states to $\mathcal{S}_{ad}$ is surjective. $\mathcal{S}_{ad}$ has a nice convex property, while the space of quasi-free states may not be convex. Therefore it is convenient to work on $\mathcal{S}_{ad}$ and then lift matrices in $\mathcal{S}_{ad}$ to associated quasi-free states. The lifting procedure is given in Lemma 4.20 Appendix 4.6.

For a pair of functions $(\Gamma(t), \Lambda(t))$, when we say an associated state $\omega(t)$, it could be any state whose two-particle correlation functions are $(\Gamma(t), \Lambda(t))$. In the case that such a state does not exist, we only use it as a notation. With the prefix quasi-free, the state refers to the associated quasi-free state shown in Lemma 4.20.

Let us work on quasi-free initial data $\omega_0$, which is lifted from a matrix in $\mathcal{S}_{ad}$. The Bogoliubov-de Gennes equations are an approximation scheme to the many-body Schrödinger equation, which are defined for two-particle correlation functions

$$i\,\partial_t \Gamma(t) = [-\Delta, \Gamma(t)] + \underbrace{[v * \rho_{\Gamma(t)}, \Gamma(t)] - [\Gamma(t), \Gamma(t)]_v + [\Lambda(t), \Lambda^*(t)]_v}_{F_1(t;v)} \tag{2.39}$$

$$i\,\partial_t \Lambda(t) = [-\Delta, \Lambda(t)]_+ + (v\Lambda)(t) + \underbrace{[v * \rho_{\Gamma(t)}, \Lambda(t)]_+ - [\Gamma(t), \Lambda(t)]_{v,+} - [\Lambda(t), \bar{\Gamma}(t)]_{v,+}}_{F_2(t;v)}, \tag{2.40}$$

where $\rho_{\Gamma(t)}(x) = \Gamma(t,x,x)$ and

$$(vA)(t,x,y) := v(x-y)A(t,x,y), \tag{2.41}$$

$$[A,B]_v := (vA)B - A(vB), \quad [A,B]_{v,+} := (vA)B + A(vB). \tag{2.42}$$

The kernel form of Equation (2.39) and (2.40) is

$$\begin{aligned} i\,\partial_t\Gamma(t,x,y) = &\left(-\Delta_x + \Delta_y\right)\Gamma(t,x,y) \\ &+ \int_{\mathbb{R}^3} dz\ \left(v(x-z) - v(y-z)\right)\Big(\Lambda(t,x,z)\bar{\Lambda}(t,y,z) \\ &-\Gamma(t,x,z)\Gamma(t,z,y) + \Gamma(t,x,y)\rho_\Gamma(z)\Big) \end{aligned} \tag{2.43}$$

$$\begin{aligned} i\,\partial_t\Lambda(t,x,y) = &\left(-\Delta_x - \Delta_y + v(x-y)\right)\Lambda(t,x,y) \\ &+ \int_{\mathbb{R}^3} dz\ \left(v(x-z) + v(y-z)\right)\big(\rho_\Gamma(z)\Lambda(t,x,y) \\ &-\Gamma(t,x,z)\Lambda(t,z,y) - \Lambda(t,x,z)\bar{\Gamma}(t,z,y)\big). \end{aligned} \tag{2.44}$$

where $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^3$.

Recently, [BSS18] formulated the Dirac–Frenkel approximation principle in terms of reduced density matrices and applied it to the Fermionic system. They obtained the Bogoliubov-de Gennes equations as an approximation to the many-body Schrödinger equation and the approximation is optimal within the class of pure quasifree states. The idea can be extended to mixed states: we project the evolution equation (2.31) onto the space of quasifree states, use the defining properties (2.34) and obtain the Bogoliubov-de Gennes equations for mixed states. When the state is pure, there is also another way of deriving the Bogoliubov-de Gennes equations as effective equations following [GM13, GM17]. We refer to Section 4.2 for details.

The total energy of $(\Gamma(t), \Lambda(t))$ (the associated state is $\omega(t)$) is defined as

$$\begin{aligned}\mathcal{E}_{BG}(\omega(t); v) :=& \operatorname{Tr}(-\Delta\Gamma(t)) + \frac{1}{2}\operatorname{Tr}\left((\rho_{\Gamma(t)} * v)\Gamma(t)\right) \\ & - \frac{1}{2}\operatorname{Tr}\left((v\Gamma)(t)\Gamma^*(t)\right) + \frac{1}{2}\operatorname{Tr}\left((v\Lambda)(t)\Lambda^*(t)\right)\end{aligned} \tag{2.45}$$

where the trace Tr is taken over $L^2(\mathbb{R}^3)$. The expression of the total energy can be derived from $\operatorname{Tr}_{\mathcal{F}_a}\left(\hat{H}\omega(t)\right)$ when $\omega(t)$ is quasi-free. A formal computation shows the time derivative of $\operatorname{Tr}_{\mathcal{F}_a}\left(\hat{H}\omega(t)\right)$ is $\operatorname{Tr}_{\mathcal{F}_a}\left(\hat{H}[\hat{H}, \omega(t)]\right)$ and it vanishes using the formal cyclicity of trace. Thus the energy of the system is conserved. A rigorous proof will be given in Chapter 4. We also give the corresponding integral form of the energy

$$\begin{aligned}\mathcal{E}_{BG}(\omega(t); v) =& \int_{\mathbb{R}^3} dx \left(\sum_{j=1}^{3} \partial_{x^j}\partial_{y^j}\Gamma\right)(t, x, x) + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y) \\ & + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v(x-y)\left(-|\Gamma|^2 + |\Lambda|^2\right)(t, x, y).\end{aligned}$$

In this model, we study the local and global well-posedness of the Bogoliubov-de Gennes equations (2.39) and (2.40) for mixed states $\omega(t)$. The two-particle correlation function $\Lambda$ of a state is anti-symmetric, i.e. $\Lambda(x, y) = -\Lambda(y, x)$, because of CAR. Therefore $\Lambda$ vanishes along the diagonal, i.e. $\Lambda(x, x) = 0$. Based on this observation, we are able to apply dispersive PDE techniques and a generalization of

Morrey's inequality to deal with all singular interaction potentials in the form

$$v(x) = \frac{1}{|x|^{2-\epsilon}}, \quad x \in \mathbb{R}^3, \quad \text{where } 0 < \epsilon \le 2. \tag{2.46}$$

In order to give a uniform argument, we assume that $0 < \epsilon < 1$. For other cases $1 \le \epsilon \le 2$, some steps of our argument need modifying. When $\epsilon = 1$, $v$ is the Coulomb potential and this case has been solved by [BSS18]. For the rest of the paper, we regard $\epsilon$ as a fixed constant which belongs to $(0,1)$.

We are dealing with data defined in the spaces

**Definition 2.10.** Suppose $k \in \mathbb{N}$ and $1 \le p \le \infty$. Let $k$ be an operator on $L^2(\mathbb{R}^d)$, the Schatten-Sobolev norm is

$$\|k\|_{\mathcal{L}^{s,p}} := \mathrm{Tr}\left(|\langle\nabla\rangle^s k \langle\nabla\rangle^s|^p\right)^{1/p}.$$

When $s = 0$, $\mathcal{L}^{0,p}$ is the usual Schatten norm and it is denoted as $\mathcal{L}^p$ for simplicity. Let $u$ be a function on $\mathbb{R}^d$ and $s \ge 0$,

$$\|u\|_{W^{s,p}} := \|u\|_{L^p} + \||\nabla|^s u\|_{L^p}, \quad \|u\|_{H^s} := \|u\|_{W^{s,2}}.$$

Since we are applying dispersive PDE techniques, the solution space, which is needed for the Banach fixed point argument to Equation (2.39) and (2.40), involves the following two Strichartz norms

**Definition 2.11.** Suppose $k(t,x,y)$ is a space-time function where $(t,x,y) \in [0,T] \times$

$\mathbb{R}^3 \times \mathbb{R}^3$ and $T \in \mathbb{R}$, then the Strichartz norms involving $s$ derivatives in the space direction $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$ are defined as

$$\begin{aligned}
\|k(t,x,y)\|_{ST^s_T} &:= \sup_{q,r\in Ad} \|\langle \nabla_{x,y}\rangle^s k(t,x,y)\|_{L^q_t L^r_x L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} && (2.47)\\
&+ \sup_{q,r\in Ad} \|\langle \nabla_{x,y}\rangle^s k(t,x,y)\|_{L^q_t L^r_y L^2_x([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)};\\
\|k(t,x,y)\|_{ST^s_{\epsilon T}} &:= \sup_{q,r\in Ad_\epsilon} \|\langle \nabla_{x,y}\rangle^s k(t,x,y)\|_{L^q_t L^r_x L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} && (2.48)\\
&+ \sup_{q,r\in Ad_\epsilon} \|\langle \nabla_{x,y}\rangle^s k(t,x,y)\|_{L^q_t L^r_y L^2_x([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
&+ \sup_{q,r\in Ad_\epsilon} \|\langle \nabla_{x,y}\rangle^s k(t,x,y)\|_{L^q_t L^r_{x-y} L^2_{x+y}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)},
\end{aligned}$$

where

$$Ad_\epsilon := \left\{(q,r)\,\middle|\,\frac{1}{q}+\frac{3}{2r}=\frac{3}{4},\ q\ge\frac{4}{2-\epsilon}\right\},\quad Ad := Ad_0.$$

The norms of the solution space for our local result are

**Definition 2.12.** Let $(\Gamma(t),\Lambda(t))$ be a pair of operators on $L^2(\mathbb{R}^3)$,

$$\begin{cases}
\|\Gamma(t)\|_{N_{1T}} := \|\Gamma(t)\|_{L^\infty_t([0,T],\mathcal{L}^1)} + \|\rho_{\Gamma(t)}(x)\|_{L^1_t L^3_x([0,T]\times\mathbb{R}^3)} + \|\Gamma(t,x,y)\|_{ST^1_T}\\
\|\Lambda(t)\|_{N_{2T}} := \|\Lambda(t,x,y)\|_{ST^1_{\epsilon T}}
\end{cases}. \quad (2.49)$$

For short, let $\omega(t)$ be the associated state,

$$\|\omega(t)\|_{N_T} := \|\Gamma(t)\|_{N_{1T}} + \|\Lambda(t)\|_{N_{2T}}.$$

Next, we describe the potential $v$ using norm $\|\cdot\|_M$, which involves all quantities we are dealt in the proofs of our theorems.

**Definition 2.13.** Let $v$ be a function on $\mathbb{R}^3$, the norm $\|v\|_M$ is defined as

$$\begin{aligned}\|v\|_M &:= \|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}} + \|(v\chi_1)(x)|x|\|_{L^3} + \|\chi_1\nabla v\|_{L^{\frac{3}{3-\epsilon/2}}} + \|v\chi_2\|_{L^\infty} + \|\chi_2\nabla v\|_{L^\infty} \\ &\quad + \|\langle\nabla\rangle(v\chi_2)\|_{L^3} + \left\|\nabla(v\chi_1)|x|^{(1-\epsilon)/2}\right\|_{L^{6/5}},\end{aligned} \tag{2.50}$$

where $\chi_1$ and $\chi_2$ are cut-off functions such that

$$\operatorname{supp}(\chi_1) \subset [0,2), \quad \operatorname{supp}(\chi_2) \subset [1,\infty), \quad \chi_1 + \chi_2 = 1,$$

and $(v\chi_1)(x) = v(x)\chi_1(|x|)$ and $(v\chi_2)(x) = v(x)\chi_2(|x|)$.

This condition (2.50) includes (2.46) for $0 < \epsilon < 1$. Recall that

**Definition 2.14.** $(\Gamma(t), \Lambda(t))$ a mild solution to Equation (2.39) and (2.40) if it satisfies the integral equations

$$\begin{cases} \Gamma(t) = e^{i\Delta t}\Gamma_0 e^{-i\Delta t} - i\displaystyle\int_0^t ds\, e^{i\Delta(t-s)} F_1(s;v) e^{-i\Delta(t-s)} \\ \Lambda(t) = e^{i\Delta t}\Lambda_0 e^{i\Delta t} - i\displaystyle\int_0^t ds\, e^{i\Delta(t-s)} ((v\Lambda)(s) + F_2(s;v)) e^{i\Delta(t-s)} \end{cases} \tag{2.51}$$

Finally, the local well-posedness theorem is as follows, which is meant to deal with correlation functions which are more general than correlation functions associated with quasi-free states.

**Theorem 2.15.** *Suppose the interaction potential $v$ satisfies*

$$\|v\|_M < \infty, \quad v(x) = v(-x) \quad \textit{and} \quad v(x) \in \mathbb{R} \quad \textit{for } x \in \mathbb{R}^3,$$

*and*

$$\Gamma_0^* = \Gamma_0, \quad \Lambda_0^* = -\bar{\Lambda}_0, \quad \Gamma_0 \in \mathcal{L}^1, \quad \Gamma_0, \Lambda_0 \in H^1\left(\mathbb{R}^6\right). \tag{2.52}$$

*For sufficiently small time interval* $[0,T]$, $T \in \mathbb{R}$, *the Bogoliubov-de Gennes equations (2.39) and (2.40) with initial conditions*

$$\Gamma(t=0) = \Gamma_0 \quad \textit{and} \quad \Lambda(t=0) = \Lambda_0,$$

*have a unique mild solution* $(\Gamma(t), \Lambda(t))$ *such that* $\|\Gamma(t)\|_{N_{1T}} + \|\Lambda(t)\|_{N_{2T}} < \infty$.

Next we extend the local theory to a global result when the initial data $(\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$. In this case, using a limiting argument, we prove that the conservation law of trace holds and the solution $(\Gamma(t), \Lambda(t))$ still satisfies Condition (2.37). The norm $N_T$ is below the energy level and the convergence of smooth solutions in $N_T$ does not imply the convergence of the energy functional. However if we assume that the energy is finite initially and use the positivity of $\Gamma(t)$ and $v$, we can recover $\Gamma(t) \in \mathcal{L}^{1,1}$ and prove the conservation law of energy. Using the conserved quantities, we extend the local mild solution $(\Gamma(t), \Lambda(t))$ obtained from Theorem 2.15 to a global one.

**Theorem 2.16.** *Suppose* $\|v\|_M < \infty$,

$$v(x) = v(-x) \quad \textit{and} \quad v(x) \geq 0 \quad \textit{for} \quad x \in \mathbb{R}^3,$$

*and the initial data* $(\Gamma(t=0), \Lambda(t=0)) = (\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$ *(the associated state is* $\omega_0$*)*

*and*

$$\Gamma_0 \in \mathcal{L}^1, \quad \Gamma_0, \Lambda_0 \in H^1\left(\mathbb{R}^6\right) \quad \textit{and} \quad \mathcal{E}_{BG}(\omega_0; v) < \infty,$$

*then there is a global mild solution* $(\Gamma(t), \Lambda(t))$ *(the associated state is* $\omega(t)$*) to the Bogoliubov-de Gennes equations (2.39) and (2.40) such that*

*(i)* $\Gamma(t) \in C\left(\mathbb{R}, H^1\right) \cap L^\infty\left(\mathbb{R}, \mathcal{L}^{1,1}\right)$ *and* $\Lambda(t) \in C\left(\mathbb{R}, H^1\right)$*;*

*(ii)* $(\Gamma(t), \Lambda(t)) \in \mathcal{S}_{ad}$ *for* $t \in \mathbb{R}$*;*

*(iii)* $Tr(\Gamma(t)) = Tr(\Gamma_0)$ *for* $t \in \mathbb{R}$ *(conservation of trace);*

*(iv)* $\mathcal{E}_{BG}(\omega(t); v) = \mathcal{E}_{BG}(\omega_0; v)$ *for* $t \in \mathbb{R}$ *(conservation of energy).*

*Remark* 2.17. $(\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$ implies that $\Gamma_0^* = \Gamma_0$, $\Lambda_0^* = -\bar{\Lambda}_0$ and $\Gamma_0$ is positive.

# Chapter 3: Hartree Equation With Constant Magnetic Field: Well-Posedness Theory

The chapter is organized in the following way: in Section 3.1 we define most notations used in the chapter; in Section 3.2 we discuss the propagator $e^{-iHt}$ and the spectral structure of $H$; in Section 3.3 we establish the collapsing estimate Theorem 2.6; in Section 3.4 we first give a low regularity result for Equation (2.12) to show that the "forcing" term $[\rho_Q * v, \bar{\Pi}_\phi]$ in Equation (2.15) is a challenging term to handle and then prove Theorem 2.3; in Section 3.5, we pose open problems for future study. In the appendix, in Section 3.6.1, we give a short review of the Heisenberg group; in Section 3.6.2 we present two families of stationary solutions to Equation (2.12); in Section 3.6.4, we show the global well-posedness of Equation (2.9) for the case when $\Gamma_0$ is of trace class and $V(x) = \frac{1}{|x|}$.

## 3.1 Preliminary

For the reader's convenience, we define most notations used in the chapter in this section.

Let $\Omega$ denote the canonical symplectic form on $\mathbb{R}^2$,

$$\Omega(x,y) := x^1 y^2 - x^2 y^1, \quad x,y \in \mathbb{R}^2, \tag{3.1}$$

and $I, J$ be matrices

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

The inner product on $L^2(\mathbb{R}^d)$ is defined to be complex linear in the first variable in this chapter, which is different the other chapters

$$\langle f, g \rangle_{L^2(\mathbb{R}^d)} : \int_{\mathbb{R}^d} f(x)\bar{g}(x)\,dx \tag{3.3}$$

.

Let $a$ and $a^\dagger$ be annihilation and creation operators

$$a := \frac{x + b\partial_x}{\sqrt{2}}, \qquad a^\dagger := \frac{x - b\partial_x}{\sqrt{2}}, \quad x \in \mathbb{R}. \tag{3.4}$$

Denote normalized Hermite polynomials by $h_j$, $j \in \mathbb{N}$,

$$h_j(x) := \frac{(a^\dagger)^j}{(b\pi)^{1/4}\sqrt{j!\,b^j}} e^{-\frac{x^2}{2b}}, \quad x \in \mathbb{R}. \tag{3.5}$$

They satisfy $\langle h_j, h_k \rangle_{L^2(\mathbb{R})} = \delta_{jk}$. $H_h$ denotes the Hermite operator

$$H_h := -\Delta_x + \frac{b^2|x|^2}{4}, \quad x \in \mathbb{R}^2. \tag{3.6}$$

We use the following tools from the harmonic analysis in the phase space [Fol89]. Let $f \in L^2(\mathbb{R})$, the Heisenberg representation $\beta$ on $L^2(\mathbb{R})$ is defined as

$$\beta(p,q,t)f := e^{i(p\hat{P}+q\hat{X}+tb)}f = e^{iqx+\frac{ibpq}{2}+ibt}f(x+pb) \tag{3.7}$$

where $x,p,q,t \in \mathbb{R}$, $\hat{P} = -ib\partial_x$ and $\hat{X}$ denotes the multiplication by $x$. For simplicity, denote $\beta(p,q,0)f$ as $\beta(p,q)f$. $\beta$ is a unitary representation.

The twisted convolution between two functions $f,g$ is

$$(f \natural g)(x) := \int_{\mathbb{R}^2} f(x-y)g(y)e^{\frac{ib}{2}\Omega(x,y)}\,dy, \tag{3.8}$$

and the "complex conjugate" $\bar{\natural}$ is defined as

$$(f\bar{\natural}g)(x) := \int_{\mathbb{R}^2} f(x-y)g(y)e^{-\frac{ib}{2}\Omega(x,y)}\,dy. \tag{3.9}$$

The Fourier-Wigner transform $V$ is defined as the matrix coefficient of the Heisenberg representation

$$V(f,g)(p,q) := \langle\beta(p,q)f,g\rangle_{L^2(\mathbb{R})} \tag{3.10}$$

$$= \int_{\mathbb{R}} e^{iqx+\frac{ibpq}{2}}f(x+pb)\bar{g}(x)\,dx \tag{3.11}$$

where $p,q \in \mathbb{R}$ and the Wigner transform $W$ is the Fourier transform of $V$

$$W(f,g)(\xi,x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} V(f,g)(p,q)e^{-i\xi p-ixq}\,dpdq \quad \xi,x \in \mathbb{R}. \tag{3.12}$$

*Remark* 3.1. All these concepts can be defined similarly in higher dimensions.

## 3.2 Properties of $H$

In this section, we discuss the one parameter unitary subgroup $e^{-iHt}$ generated by $-iH$, where

$$H = D^* D = -\partial_{x^1}^2 - \partial_{x^2}^2 - ib\left(x^2\partial_{x^1} - x^1\partial_{x^2}\right) + \frac{b^2}{4}\left(|x^1|^2 + |x^2|^2\right) - b, \tag{3.13}$$

$b > 0$ and the spectral structure of $H$. They are crucial ingredients for the collapsing estimate. The formula for $e^{-iHt}$ is derived by applying the metaplectic representation.

**Theorem 3.2.** *Given the Schrödinger equation*

$$i\,\partial_t f(t,x) = Hf(t,x), \quad f(0,x) = f_0(x), \tag{3.14}$$

*where* $f_0(x) \in \mathcal{S}(\mathbb{R}^2)$*, the formula for the solution is*

$$\left(e^{-iHt}f_0\right)(x) = \begin{cases} \dfrac{b\,e^{ibt}}{4\pi i\sin(bt)}\displaystyle\int_{\mathbb{R}^2}\exp\left(\frac{ib(x-y)^2}{4\tan(bt)} - \frac{ib}{2}\Omega(x,y)\right)f_0(y)\,dy, & t \neq \frac{\pi}{b}k \\ f_0(x), & t = \frac{\pi}{b}k \end{cases} \tag{3.15}$$

*where* $k \in \mathbb{Z}$.

*Proof.* Consider the metaplectic representation $\mu$ [Fol89, Chapter 4] from the metaplectic group $Mp(4,\mathbb{R})$ to the unitary group $U\left(L^2(\mathbb{R}^2)\right)$. The corresponding infinitesimal representation $d\mu$ is

$$d\mu : \mathfrak{sp}(4,\mathbb{R}) \to \mathfrak{u}\left(L^2(\mathbb{R}^2)\right) \tag{3.16}$$

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto -\frac{1}{2i}\begin{pmatrix} \hat{Q} & \hat{P} \end{pmatrix}\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix}, \tag{3.17}$$

where $\hat{Q} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, $\hat{P} = \begin{pmatrix} -i\partial_{x^1} \\ -i\partial_{x^2} \end{pmatrix}$ and $A^t$ denotes the transpose matrix of $A$. Under $d\mu$, $-i\,(H+b) \in \mathfrak{u}\left(L^2(\mathbb{R}^2)\right)$ corresponds to

$$\mathcal{A} = \begin{pmatrix} bJ & \frac{b^2}{2}\,I \\ -2\,I & bJ \end{pmatrix} \in \mathfrak{sp}(4,\mathbb{R}).$$

In order to apply Theorem 3.28 from the appendix to get the integral Formula (3.15), we need to compute the explicit form for the one parameter subgroup $\exp(\mathcal{A}t)$ in the symplectic group $Sp(4,\mathbb{R})$. Since $\mathcal{A}$ can be written as a sum of two commuting matrices

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \frac{b^2}{2}\,I \\ -2\,I & 0 \end{pmatrix},$$

then

$$
\begin{aligned}
&\exp(\mathcal{A}t) \\
&= \exp\left(\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} bt\right) \cdot \exp\begin{pmatrix} 0 & \frac{b^2 t}{2} I \\ -2t\, I & 0 \end{pmatrix} \\
&= \begin{pmatrix} \exp\left(Jbt\right) & 0 \\ 0 & \exp\left(Jbt\right) \end{pmatrix} \cdot \begin{pmatrix} \cos(bt)\, I & \frac{b}{2}\sin(bt)\, I \\ -\frac{2}{b}\sin(bt)\, I & \cos(bt)\, I \end{pmatrix}.
\end{aligned}
$$

Using Theorem 3.28, we have the expression

$$
\left(\mu(\exp(\mathcal{A}t))f_0\right)(x) = \frac{1}{2\pi\cos(bt)} \int_{\mathbb{R}^2} \exp\left(-iS(t,x,\xi)\right) \hat{f}_0(-\xi)\, d\xi, \tag{3.18}
$$

where the phase function $S$ is

$$
S(t,x,\xi) = \frac{\tan(bt)}{b}\,|\xi|^2 + x\xi + \tan(bt)\Omega(x,\xi) + \frac{b\tan(bt)}{4}|x|^2
$$

and $\xi = (\xi^1, \xi^2)$, $x = (x^1, x^2)$. To obtain Formula (3.15),

$$
\begin{aligned}
&\left(\mu(\exp(\mathcal{A}t)) f_0\right)(x) \\
=&\frac{1}{(2\pi)^2 \cos(bt)} \int_{\mathbb{R}^2} f_0(y)\, dy \int_{\mathbb{R}^2} \exp\left(-iS(x,\xi) + iy\xi\right) d\xi \\
=&\frac{1}{(2\pi)^2 \cos(bt)} \int_{\mathbb{R}^2} f_0(y)\, dy \int_{\mathbb{R}^2} \exp\left(\frac{bi}{4\tan(bt)} \left(x - \tan(bt)Jx - y\right)^2\right) \\
&\cdot \exp\left(-i\left[\frac{b\tan(bt)}{4}|x|^2 + \frac{\tan(bt)}{b}\left(\xi + \frac{b}{2\tan(bt)}(x - \tan(bt)Jx - y)\right)^2\right]\right) d\xi \\
=&\frac{b}{4\pi i \sin(bt)} \int_{\mathbb{R}^2} \exp\left(\frac{ib}{4\tan(bt)}(x-y)^2 - \frac{ib}{2}\Omega(x,y)\right) f_0(y)\, dy.
\end{aligned} \tag{3.19}
$$

Let us denote (3.19) by $sol(t)f_0$.

Theorem 3.28 is valid as long as the matrix $\cos(bt)I$ is not degenerate. Since $\cos(bt)$ vanishes at $\frac{\pi}{2b}$, we only obtain the Formula (3.15) for $t \in \left[0, \frac{\pi}{2b}\right)$. Next we show that Formula (3.19) is valid on $\mathbb{R}$. Formula (3.19) is defined when $t \in (0, \pi/b)$. By direct computation,

$$
sol(t+s)f_0 = sol(t)sol(s)f_0, \quad \text{for } t > 0,\ s > 0,\ t + s < \frac{\pi}{b},
$$

i.e. $sol(t)$ is a semigroup when $t \in (0, \pi/b)$. Besides $sol(t)$ is also continuous with respect to the strong operator topology when $t \in [0, \pi/b)$. This is because when $t \in [0, \pi/2b)$, we obtain Formula (3.19) by the metaplectic representation; when $t \in [\pi/2b, \pi/b)$, $sol(t) = sol(\pi/2b)sol(t - \pi/2b)$. Therefore, by the uniqueness of the one parameter unitary subgroup generated by $d\mu(\mathcal{A})$, $e^{-i(H+b)t} = sol(t)$ is true for $t \in [0, \pi/b)$. As $t \to \pi/b$, from (3.18), we see that the phase function $S(t, x, \xi) \to x\xi$

and $\mu(\exp(\mathcal{A}t))\, f_0 \to -f_0$ pointwise. By the dominant convergence theorem, $sol(t)f_0$ also converges to $-f_0$ in $L^2(\mathbb{R}^2)$. In summary, we have obtained Formula (3.15) for $t \in [0, \pi/b]$ and showed that $e^{-iHt}$ is of period $\pi/b$. Therefore $e^{-i(H+b)t} = sol(t)$ holds for $t \in \mathbb{R}$. □

*Remark* 3.3. According to the metaplectic representation $\mu$, one can also conclude that $e^{-i(H+b)\pi/b} = -1$ by the observation that $exp(\mathcal{A}t) : [0, \pi/b] \to Sp(4, \mathbb{R})$ is the generator of the fundamental group $\pi_1(Sp(4,\mathbb{R}))$ of $Sp(4,\mathbb{R})$ and the metapletic group is the double cover of $Sp(4,\mathbb{R})$.

Based on the formula (3.15) and the machinery in [GV92], we obtain the Strichartz estimate to arbitrary finite time.

**Corollary 3.4.** *Fix any time* $T \in \mathbb{R}$*,*

$$\left\|e^{-iHt} f\right\|_{L_t^q L_x^r([0,T]\times\mathbb{R}^2)} \lesssim_{q,r,T} \|f\|_{L^2(\mathbb{R}^2)}, \tag{3.20}$$

*where* $(q, r)$ *satisfies (2.25).*

*Proof.* For the time being, let $T$ be a fixed time. There is a positive integer $n$ such that $T_\epsilon = \frac{T}{n} \leq \frac{\pi}{10b}$. According to Theorem 3.2, for $t < T_\epsilon$,

$$\left\|e^{-iHt} f\right\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{t} \|f\|_{L^1(\mathbb{R}^2)}.$$

Since $e^{-iHt}$ is unitary, by [GV92], $\|e^{-iHt} f\|_{L_t^q L_x^r([0,T_\epsilon]\times\mathbb{R}^2)} \lesssim_{q,r} \|f\|_{L^2(\mathbb{R}^2)}$, where $(q, r)$ satisfies (2.25). For any integer $j$, $1 \leq j \leq n$, repeat the above argument on the time interval $[(j-1)T_\epsilon, jT_\epsilon]$,

$$\left\|e^{-iHt}f\right\|_{L_t^q L_x^2([(j-1)T_\epsilon, jT_\epsilon]\times\mathbb{R}^2)} \lesssim_{q,r} \left\|e^{-iH(j-1)T_\epsilon}f\right\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)}.$$

By virtue of the Minkowski inequality

$$\left\|e^{-iHt}f\right\|_{L_t^q L_x^r([0,nT_\epsilon]\times\mathbb{R}^2)} \le \sum_{j=1}^{n}\left\|e^{-iHt}f\right\|_{L_t^q L_x^2([(j-1)T_\epsilon, jT_\epsilon]\times\mathbb{R}^2)} \lesssim_{q,r} n\|f\|_{L^2(\mathbb{R}^2)}.$$

□

The spectrum of $H$ is well-known in the physics literature. Here we give a discussion of its spectral structure and some formulas based on the Fourier-Wigner transform. $H$ is a semi-positive self-adjoint operator on $L^2(\mathbb{R}^2)$. Since for any $f \in \mathcal{D}(D)$,

$$Df = \left(-2\partial_{\bar{z}} - \frac{b}{2}z\right)f = e^{-b|z|^2/4}\left(-2\partial_{\bar{z}}\right)\left(e^{b|z|^2/4}f\right),$$

and $\partial_{\bar{z}}$ is elliptic, the null space $\mathcal{H}_0$ of $H$ consists of all functions in the form $g(z)e^{-b|z|^2/4}$, where $g(z)$ is an entire function. To rephrase it, $e^{b|z|^2/4}\mathcal{H}_0$ is a Fock-Bargmann space [Fol89, Section 1.6] with probability measure $b\,e^{-b|z|^2/2}d\mu/2\pi$, where $d\mu$ is the Lebesgue measure on $\mathbb{C}$. Thus, with respect to the canonical Hermitian inner product on $L^2(\mathbb{R}^2)$, $\mathcal{H}_0$ has an orthonormal basis

$$e_{0j}(z) := \frac{z^j}{\sqrt{\pi j!(2/b)^{j+1}}}\exp\left(-\frac{b|z|^2}{4}\right), \tag{3.21}$$

where $z \in \mathbb{C} \simeq \mathbb{R}^2,\ j \in \mathbb{N}$, and the integral kernel $P_0(x,y)$ associated to the projection

$P_0 : L^2(\mathbb{R}^2) \to \mathcal{H}_0$ is

$$P_0(x,y) = \frac{b}{2\pi}\exp\left(-\frac{b|x-y|^2}{4} - \frac{ib}{2}\,\Omega(x,y)\right) \tag{3.22}$$
$$= \frac{b}{2\pi}\exp\left(-\frac{b}{4}\left(|z_x|^2 + |z_y|^2\right) + \frac{b}{2}z_x\bar{z}_y\right),$$

where $z_x = x^1 + ix^2$, $x = (x^1, x^2) \in \mathbb{R}^2$, $z_y = y^1 + iy^2$ and $y = (y^1, y^2) \in \mathbb{R}^2$.

Using the commutation relation $[D, (D^*)^k] = 2bk(D^*)^{k-1}$, we obtain other eigenspaces $\mathcal{H}_k = (D^*)^k(\mathcal{H}_0)$ associated to eigenvalue $2bk$ and orthonormal bases of $\mathcal{H}_k$ for $k \in \mathbb{N}$,

$$e_{kj}(z) := \frac{(D^*)^k}{\sqrt{(2b)^k k!}} e_{0j}(z), \quad j \in \mathbb{N}. \tag{3.23}$$

On the level of eigenspaces, $H$ has a ladder operator structure $D^*\mathcal{H}_k = \mathcal{H}_{k+1}$ and $D(\mathcal{H}_k) = \mathcal{H}_{k-1}$. Therefore we call $D$ and $D^*$ annihilation and creation operators respectively. Furthermore,

**Lemma 3.5.** *The space $L^2(\mathbb{R}^2)$ is decomposed orthogonally as follows*

$$L^2(\mathbb{R}^2) = \overline{\bigoplus_{k\in\mathbb{N}} \mathcal{H}_k},$$

*which implies that $H$ has a discrete spectrum $\sigma(H) = \{2bk\}_{k=0}^{\infty}$ with corresponding eigenspaces $\mathcal{H}_k$.*

*Proof.* Consider the related Hermite operator $H_h$, $x = (x^1, x^2) \in \mathbb{R}^2$, and associated creation and annihilation operators

$$a_j^\dagger = \partial_{x^j} - \frac{b}{2}x^j,\ a_j = -\partial_{x^j} - \frac{b}{2}x^j, \quad j \in \{1,2\}.$$

$\left\{\left(a_1^\dagger\right)^j \left(a_2^\dagger\right)^l e^{-b|x|^2/4}\right\}_{j,l\in\mathbb{N}}$ is a basis for $L^2(\mathbb{R}^2)$. Since

$$\begin{aligned}\left(a_1^\dagger\right)^j \left(a_2^\dagger\right)^l e^{-b|x|^2/4} &= e^{b|x|^2/4}(\partial_{x^1})^j(\partial_{x^2})^l e^{-b|x|^2/2}\\ &= i^l\, e^{b|x|^2/4}(\partial_z + \partial_{\bar z})^j(\partial_z - \partial_{\bar z})^l e^{-b|x|^2/2},\end{aligned}$$

and bases of $\mathcal{H}_k$ are in the form

$$(D^*)^k\left(z^j e^{-b|z|^2/4}\right) = e^{b|z|^2/4}(2\partial_z)^k\left(-\frac{2}{b}\partial_{\bar z}\right)^j e^{-b|z|^2/2},$$

$\left(a_1^\dagger\right)^j \left(a_2^\dagger\right)^l e^{-b|x|^2/4}$ can be written as a linear combination of bases of $\mathcal{H}_k$. Therefore the $L^2$-closure of $\bigoplus_{k\in\mathbb{N}}\mathcal{H}_k$ is $L^2(\mathbb{R}^2)$. $\square$

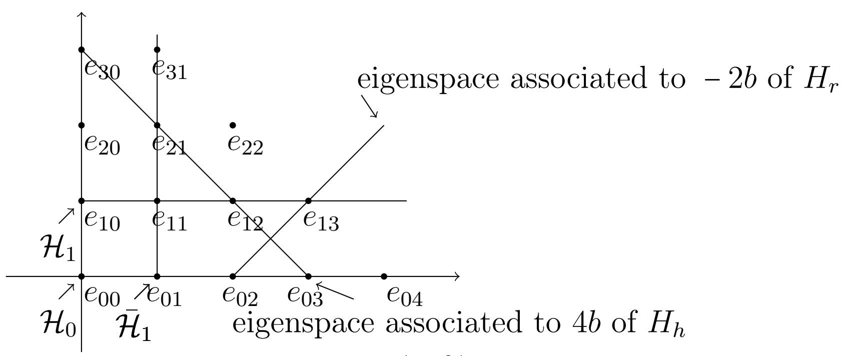
We can also derive the spectrum of $H$ by first decomposing $H$ as a sum of three operators: the constant operator $-b$, the Hermite operator $H_h$ and the rotation vector field $H_r = -ib(x^2\partial_{x^1} - x^1\partial_{x^2}) = \bar z\partial_{\bar z} - z\partial_z$, i.e.

$$H = H_h + H_r - b. \tag{3.24}$$

The three operators all commute with each other and they all share same eigenvectors. More precisely,

$$H_h e_{kj} = (k + j + 1)b\, e_{kj}, \quad H_r e_{kj} = (k - j)b\, e_{kj}.$$

Then $H e_{kj} = (H_h + \bar{z}\partial_{\bar{z}} - z\partial_z - b)\, e_{kj} = 2kb\, e_{kj}$. Displaying all eigenvectors $e_{kj}$ schematically in Figure 3.1, all rows correspond to different eigenspaces of $H$, all columns correspond to different eigenspaces of the complex conjugate $\bar{H}$, all lines with slope equal to $-1$ correspond to different eigenspaces of $H_h$ and all lines slope equal to 1 correspond to different eigenspaces of $H_r$.




$e_{kj}$ are in the form: $e_{kj}(z) = \begin{cases} z^{j-k} p\left(\frac{b|z|^2}{2}\right) & j \geq k, \\ \bar{z}^{k-j} p\left(\frac{b|z|^2}{2}\right) & k > j, \end{cases}$ where $p$ is a polynomial of degree $\min\{j,k\}$.

Figure 3.1: Canonical Eigenfunctions of $H$

Next we find projection kernels for $P_k$ using $P_0(x,y)$ and the ladder structure $\mathcal{H}_k = (D^*)^k \mathcal{H}_0$. They can be expressed in terms of the Fourier-Wigner transform.

**Lemma 3.6.** *The projection $P_k$ associated to the eigenspace $\mathcal{H}_k$ can be expressed as*

$$P_k f = \frac{b}{2\pi} V(h_k, h_k) \natural f, \tag{3.25}$$

*where $f \in L^2(\mathbb{R}^2)$. More explicitly, the integral kernel $P_k(x,y)$ of $P_k$ is*

$$P_k(x,y) = \frac{b}{2\pi} L_k\left(\frac{b|x-y|^2}{2}\right) \exp\left(-\frac{b|x-y|^2}{4} - \frac{ib}{2}\,\Omega(x,y)\right), \tag{3.26}$$

*where $x, y \in \mathbb{R}^2$ and $L_k(\lambda) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-\lambda)^j}{j!}$, $(\lambda \in \mathbb{R})$ are Laguerre polynomials.*

*Proof.* Suppose $g \in \mathcal{S}(\mathbb{R})$, the Fourier-Wigner transform of $g$ and $e^{-\lambda^2/2b}$ is

$$\begin{aligned} V\left(g, e^{-\lambda^2/2b}\right)(x^1, x^2) &= \int_{\mathbb{R}} e^{ix^2\lambda + ibx^1x^2/2} g(\lambda + bx^1) e^{-\lambda^2/2b}\, d\lambda \\ &= e^{-b|z|^2/4} \int_{\mathbb{R}} e^{\lambda z - \lambda^2/2b - bz^2/4} g(\lambda)\, d\lambda, \end{aligned}$$

where $z = x^1 + i\,x^2$. The following map

$$\left(g \mapsto \int_{\mathbb{R}} e^{\lambda z - \lambda^2/2b - bz^2/4} g(\lambda)\, d\lambda\right)$$

defines a Bargmann transform from $L^2(\mathbb{R})$ to the Fock-Bargmann space with weight $e^{-b|z|^2/2} d\mu$. Since the correspondence is isomorphic, we identify $L^2(\mathbb{R})$ with $\mathcal{H}_0$. Note that $D^*$ and the creation operator $a^\dagger$ are connected through the identity

$$\frac{D^*}{\sqrt{2}} V(g, e^{-\lambda^2/2b}) = V\left(g, a^\dagger e^{-\lambda^2/2b}\right). \tag{3.27}$$

Then $L^2(\mathbb{R})$ corresponds to $\mathcal{H}_k = (D^*)^k \mathcal{H}_0$ through

$$\left(g \mapsto V\left(g, \left(a^\dagger\right)^k e^{-\lambda^2/2b}\right)\right).$$

Therefore for any $f \in L^2(\mathbb{R}^2)$, there is a sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ such that

$$f(x) = \sum_{k=0}^{\infty} V\left(f_k, \frac{(a^\dagger)^k}{(\pi b)^{1/4}\sqrt{k!(2b)^k}} e^{-\lambda^2/2b}\right) = \sum_{k=0}^{\infty} V\left(f_k, h_k\right).$$

Using Lemma 3.25 and Theorem 3.26 from the appendix 3.6,

$$\begin{aligned}
&\overline{V\left(h_j, h_j\right)} \natural \overline{f(x)} = \sum_{k=0}^{\infty} \frac{2\pi}{b} \left\langle h_k, h_j \right\rangle \overline{V\left(f_k, h_j\right)} \\
&= \frac{2\pi}{b} \sum_{k=0}^{\infty} \delta_{jk} \overline{V\left(f_k, h_j\right)} = \frac{2\pi}{b} \overline{V(f_j, h_j)} \\
&\Longrightarrow \begin{cases} P_k = \dfrac{b}{2\pi} V\left(h_k, h_k\right) \overline{\natural} \quad \text{or} \\ P_k(x,y) = \dfrac{b}{2\pi} L_k\left(\dfrac{b}{2}|x-y|^2\right) \exp\left(-\dfrac{b|x-y|^2}{4} - \dfrac{ib\,\Omega(x,y)}{2}\right). \end{cases}
\end{aligned}$$

□

*Remark* 3.7. Similarly for $\bar{H}$, the projection $\bar{P}_k$ onto the $k$-th eigenspace of $\bar{H}$ is

$$\bar{P}_k f = \frac{b}{2\pi} V(h_k, h_k) \natural f, \quad f \in L^2(\mathbb{R}^2), \tag{3.28}$$

and the integral kernel of $\bar{P}_k$ is simply the complex conjugation of $P_k(x,y)$.

*Remark* 3.8. $D$ commutes with complex conjugates $\bar{D}$ and $\bar{D}^*$.

At the end of this section, we list some results about $H$ for later use.

The difference between $H^{1/2}$ and $D$ can be analogous to the difference between $(-\Delta)^{1/2}$ and $\nabla$. Generally for any $f \in \mathcal{D}(H^{1/2})$, $H^{1/2} f$ is not the same as $Df$. It is apparent when decompose $f$ as $f = \sum_{k=0}^{\infty} P_k f$ and apply $H^{1/2}$ and $D$ to $f$ separately

$$H^{1/2} f = \sum_{k=1}^{\infty} (2bk)^{1/2} P_k f, \quad Df = \sum_{k=1}^{\infty} DP_k f,$$

where $(2bk)^{1/2} P_k f$ and $DP_k f$ are in $\mathcal{H}_k$ and $\mathcal{H}_{k-1}$ respectively. However, they have the same $L^2$ norms

$$\left\langle H^{1/2} f, H^{1/2} f \right\rangle_{L^2(\mathbb{R}^2)} = \langle Hf, f \rangle_{L^2(\mathbb{R}^2)} = \langle D^* D f, f \rangle_{L^2(\mathbb{R}^2)} = \langle Df, Df \rangle_{L^2(\mathbb{R}^2)} . \tag{3.29}$$

More generally, for $1 < p < \infty$,

$$\| (H + b)^{1/2} f \|_{L^p(\mathbb{R}^2)} \sim_p \| Df \|_{L^p(\mathbb{R}^2)} + \| D^* f \|_{L^p(\mathbb{R}^2)} . \tag{3.30}$$

*Remark* 3.9. To see why (3.30) is true, note that our vector field potential $A$ satisfies $A \in L^2_{loc}(\mathbb{R}^3)^3$ and the magnetic field $\mathbf{B} = (0, 0, b)$ is constant. Then by [BA10, Theorem 1.3, 1.6], for $1 < p < \infty$,

$$\| Lf \|_{L^p(\mathbb{R}^3)} \sim_p \left\| \left( H - \partial_{x^3}^2 + b \right)^{1/2} f \right\|_{L^p(\mathbb{R}^3)} ,$$

where $L = \left( -i\partial_{x^1} + \frac{b}{2} x^2, -i\partial_{x^2} - \frac{b}{2} x^1, -i\partial_{x^3} \right)$ and $x = (x^1, x^2, x^3)$. Since in the third dimension it is known that $\| \partial_{x^3} g \|_{L^p(\mathbb{R})} \sim_p \| (-\partial_{x^3}^2)^{1/2} g \|_{L^p(\mathbb{R})}$ and

$$\left\| \left( -i\partial_{x^1} + \frac{b}{2} x^2, -i\partial_{x^2} - \frac{b}{2} x^1 \right) f \right\|_{L^p(\mathbb{R}^2)} \sim \| Df \|_{L^p(\mathbb{R}^2)} + \| D^* f \|_{L^p(\mathbb{R}^2)} ,$$

we obtain (3.30).

Unlike $(-\Delta)^{1/2}$ and $\nabla$, where they both commute with $\Delta$, $[D, H] = 2bD$.

There is no comparison between $\|\nabla f\|_{L^2}$ and $\|Df\|_{L^2}$. For example,

$$\|De_{0k}\|_{L^2(\mathbb{R}^2)} = 0,$$

for any $k \in \mathbb{N}$. But

$$\|\nabla e_{0k}\|_{L^2(\mathbb{R}^2)} = \|2\partial_{\bar{z}} e_{0k}\|_{L^2(\mathbb{R}^2)} = \frac{\sqrt{(k+1)b}}{\sqrt{2}} \|e_{0k+1}\|_{L^2(\mathbb{R}^2)} = \frac{\sqrt{(k+1)b}}{\sqrt{2}}$$

blows up as $k$ approaches infinity. On the other hand, taking $f \in C_c^\infty(\mathbb{R}^2)$, consider the translation $f_{\tilde{x}} = f(x - \tilde{x})$, then $\|\nabla f_{\tilde{x}}\|_{L^2(\mathbb{R}^2)} = \|\nabla f\|_{L^2(\mathbb{R}^2)}$. But

$$\|Df_{\tilde{x}}\|_{L^2(\mathbb{R}^2)} \to \infty \quad \text{as} \quad \tilde{x} \to \infty.$$

However there is a pointwise identity, for $f, g \in \mathcal{S}(\mathbb{R}^2)$,

$$-2\partial_{\bar{z}}(f\bar{g}) = (Df)\bar{g} - f\overline{D^* g},$$

which implies

$$|\partial_{\bar{z}}|f|| = \left|\frac{\partial_{\bar{z}}\left(f\bar{f}\right)}{2|f|}\right| \le \frac{\left|(Df)\,\bar{f}\right|}{|2f|} + \frac{\left|f\overline{D^* f}\right|}{2|f|} = \frac{1}{2}\left(|Df| + |D^* f|\right), \tag{3.31}$$

i.e. $|\partial_{\bar{z}}|f|| \lesssim |Df| + |D^* f|$. Based on this pointwise inequality, we generalize the

Sobolev inequality to cases involving $H$.

**Lemma 3.10.** *For $2 \le q < \infty$,*

$$\|f\|_{L^q(\mathbb{R}^2)} \lesssim_q \left\| \langle H \rangle^{1/2} f \right\|_{L^2(\mathbb{R}^2)}. \tag{3.32}$$

*Proof.* Use the pointwise inequality (3.31),

$$\begin{aligned} \|\nabla |f|\|_{L^2(\mathbb{R}^2)} = \|-2\partial_{\bar{z}}|f|\|_{L^2(\mathbb{R}^2)} &\le \|Df\|_{L^2(\mathbb{R}^2)} + \|D^* f\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|Df\|_{L^2(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

and apply the usual $n$-endpoint Sobolev inequality,

$$\begin{aligned} \|f\|_{L^q(\mathbb{R}^2)} = \||f|\|_{L^q(\mathbb{R}^2)} &\lesssim_q \|f\|_{L^2(\mathbb{R}^2)} + \|\nabla |f|\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^2)} + \|Df\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

□

## 3.3 Strichartz and Collapsing Estimates

In this section, we study the linear equation $i\,\partial_t \gamma(t) = [H, \gamma(t)]$. The formula of the propagator $e^{-iHt}$ and the spectral structure of $H$ from Section 3.2 are the basic tools for our discussion. Similar to Corollary 3.4, for any finite time $T$, we obtain the Strichartz estimate for $e^{-iHt}\gamma_0 e^{iHt} = e^{-i(H_x - \bar{H}_y)t}\gamma_0$.

**Proposition 3.11.** *Let $\gamma(t, x, y) = e^{-i(H_x - \bar{H}_y)t}\gamma_0(x, y)$ be the solution to Equation*

*(2.27), then for any $T > 0$ and $s \geq 0$,*

$$\left\| \langle H_x \rangle^{s/2} \langle \bar{H}_y \rangle^{s/2} \gamma(t,x,y) \right\|_{L_t^q L_x^r L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \lesssim_{q,r,T} \left\| \langle H_x \rangle^{s/2} \langle \bar{H}_y \rangle^{s/2} \gamma_0(x,y) \right\|_{L_{x,y}^2}, \tag{3.33}$$

*and*

$$\left\| \langle H_x \rangle^{s/2} \langle \bar{H}_y \rangle^{s/2} \gamma(t,x,y) \right\|_{L_t^q L_y^r L_x^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \lesssim_{q,r,T} \left\| \langle H_x \rangle^{s/2} \langle \bar{H}_y \rangle^{s/2} \gamma_0(x,y) \right\|_{L_{x,y}^2}$$

*where $(q,r)$ satisfies (2.25) and $x, y \in \mathbb{R}^2$. Furthermore, by duality, the following dual estimate holds*

$$\left\| \int_0^T e^{i(\bar{H}_x - H_y)t} F(t,x,y)\, dt \right\|_{L_{x,y}^2} \lesssim_{q',r',T} \| F(t,x,y) \|_{L_t^{q'} L_x^{r'} L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}, \tag{3.34}$$

*and*

$$\left\| \int_0^T e^{i(\bar{H}_x - H_y)t} F(t,x,y)\, dt \right\|_{L_{x,y}^2} \lesssim_{q',r',T} \| F(t,x,y) \|_{L_t^{q'} L_y^{r'} L_x^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)},$$

*where*

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

*Proof.* The two linear estimates are symmetric with respect to $x$ and $y$, we show the estimate for one of them and the other one is obtained by swapping roles of $x$

and $y$. Apply $\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}$ to Equation (2.27),

$$i\,\partial_t\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma(t,x,y) = \left(H_x - \bar{H}_y\right)\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma(t,x,y).$$

View $e^{-i\left(H_x-\bar{H}_y\right)t}$ as a map on the Hilbert space of $L^2(\mathbb{R}^2)$-valued functions. It is unitary since the Hilbert space $\{f\,|\,f:\mathbb{R}^2\to L^2\left(\mathbb{R}^2\right)\}$ is canonically isometric to $L^2\left(\mathbb{R}^2\times\mathbb{R}^2\right)$. Besides, using Formula (3.15), for $t<T_\epsilon\le\frac{\pi}{10b}$,

$$\begin{aligned}\left\|e^{-i\left(H_x-\bar{H}_y\right)t}\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma_0\right\|_{L^\infty_x L^2_y} &\lesssim \frac{1}{t}\left\|e^{i\bar{H}_y t}\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma_0\right\|_{L^1_x L^2_y}\\ &= \frac{1}{t}\left\|\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma_0\right\|_{L^1_x L^2_y}.\end{aligned}$$

Then using the Strichartz estimate [KT98],

$$\left\|\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma(t,x,y)\right\|_{L^q_t L^r_x L^2_y([0,T_\epsilon]\times\mathbb{R}^2\times\mathbb{R}^2)} \lesssim_{q,r} \left\|\langle H_x\rangle^{s/2}\langle \bar{H}_y\rangle^{s/2}\gamma_0(x,y)\right\|_{L^2_{x,y}}.$$

Following the same patching argument as Corollary 3.4, we obtain the linear estimate. □

In order to show Theorem 2.6, we will decompose the initial data $\gamma_0(x,y)$ based on the spectral structures of $H_x$ and $\bar{H}_y$. According to Lemma 3.6, denote

$$\gamma_{jk}(x,y) := P_{xj}\bar{P}_{yk}\gamma_0 = V(h_j,h_j)\natural_x V(h_k,h_k)\underset{y}{\natural}\gamma_0, \tag{3.35}$$

and we obtain the decomposition

$$\begin{aligned}
&\gamma_0(x,y)\\
&=\sum_{j,k\in\mathbb{N}}\gamma_{jk}(x,y)\\
&=\sum_{j,k\in\mathbb{N}}\int_{\mathbb{R}^2\times\mathbb{R}^2}V(h_j,h_j)(x-\tilde{x})V(h_k,h_k)(y-\tilde{y})e^{-ib[\Omega(x,\tilde{x})-\Omega(y,\tilde{y})]/2}\gamma_{jk}(\tilde{x},\tilde{y})\,d\tilde{x}d\tilde{y},
\end{aligned} \tag{3.36}$$

where $P_{xj}(\bar{P}_{yk})$ means the projection of $\gamma_0(x,y)$ onto $\mathcal{H}_j(\bar{\mathcal{H}}_k)$ with respect to the $x(y)$ variable. Then in the kernel form, the evolution of $\gamma_0$ under Equation (2.27) can be expressed as

$$\left(e^{-(H_x-\bar{H}_y)it}\gamma_0\right)(x,y)=\sum_{j,k\in\mathbb{N}}e^{-2b(j-k)it}\gamma_{jk}(x,y). \tag{3.37}$$

When we compute the space Fourier transform of (3.36), associated Laguerre polynomials $L_n^\alpha(\lambda)$ appear in the collapsing term

$$L_n^\alpha(\lambda)=\sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-\lambda)^j}{j!},\quad \lambda\in\mathbb{R},\ n\in\mathbb{N},\ \alpha>-1. \tag{3.38}$$

Let us discuss properties of associated Laguerre polynomials, which are needed for the collapsing estimate.

**Lemma 3.12.** *For $j,n,c\in\mathbb{N}$,*

$$\frac{n!}{(n+j)!}\max_{\lambda\ge 0}\lambda^{j+c}\left(L_n^j\right)^2(\lambda)e^{-\lambda}\le 4^c(j+2n+c)^c. \tag{3.39}$$

*Furthermore, since associated Laguerre polynomials are related to $V(h_j, h_k)$ by Theorem 3.26 in Appendix 3.6.3, (3.39) is equivalent to*

$$\|\bar{w}^c V(h_j, h_k)(p,q)\|_{L^\infty(\mathbb{R}^2)} \leq \left(\frac{2}{\sqrt{b}}\right)^c (j+k+c)^{c/2}, \tag{3.40}$$

*where $j, k, c \in \mathbb{N}$, and $w = p + iq \in \mathbb{C}$.*

*Proof.* We prove (3.40) by induction on $c$. Consider the basic case $c = 0$, by Cauchy-Schwartz inequality, for $j, k \in \mathbb{N}$,

$$\begin{aligned} |V(h_j, h_k)| = |\langle \beta(p,q) h_j, h_k \rangle| &\leq \|\beta(p,q) h_j\|_{L^2} \|h_k\|_{L^2} \\ &= \|h_j\|_{L^2} \|h_k\|_{L^2} = 1. \end{aligned}$$

Assume (3.40) holds for $c = n \in \mathbb{N}$. When $c = n+1$, using the following commutation relations,

$$\left[a, a^\dagger\right] = b, \quad [a, \beta(p,q)] = -\frac{b}{\sqrt{2}} \bar{w} \beta(p,q),$$

we obtain

$$\begin{aligned} &\bar{w}^{n+1} V(h_j, h_k)(p,q) \\ &= -\frac{\sqrt{2}}{b} \bar{w}^n \langle [a, \beta(p,q)] h_j, h_k \rangle \\ &= -\frac{\sqrt{2}}{b} \left( \sqrt{(k+1)b}\, \bar{w}^n V(h_j, h_{k+1}) - \sqrt{jb}\, \bar{w}^n V(h_{j-1}, h_k) \right)(p,q). \end{aligned}$$

Using the induction assumption, $|\bar{w}^{n+1} V(h_j, h_k)(p,q)|$

$$\leq \frac{2^{n+1/2}}{b^{(n+1)/2}} \left( \sqrt{k+1}(j+k+1+n)^{n/2} + \sqrt{j}(j+k+n-1)^{n/2} \right)$$
$$\leq \left( \frac{2}{\sqrt{b}} \right)^{n+1} (j+k+n+1)^{(n+1)/2}.$$

Therefore (3.40) holds for all $c \in \mathbb{N}$. $\square$

There is a more refined estimate than Lemma 3.12 when $c = 1$,

**Theorem 3.13.** *[Kra05, Kra07] Let $n \geq 1$, $\alpha > -1$, then*

$$\frac{n!}{\mathcal{G}(n+\alpha+1)} \max_{\lambda \geq 0} \left( \lambda^{\alpha+1} e^{-\lambda} \left( L_n^{\alpha} \right)^2 (\lambda) \right) < 6 n^{1/6} \sqrt{n+\alpha+1},$$

*where $\mathcal{G}$ denotes the gamma function*

$$\mathcal{G}(z) := \int_0^{\infty} \lambda^{z-1} e^{-\lambda} \, d\lambda, \quad \mathcal{R}(z) > 0.$$

In the case $c = 1$, the upper bound in (3.39) is essentially $(j+n)$ for large $n$ and $j$. When we consider the asymptotic behavior of $\frac{n!}{(n+j)!} \max_{\lambda \geq 0} \lambda^{j+1} \left( L_n^j \right)^2 (\lambda) e^{-\lambda}$ in terms of $j$ and $n$, Krasikov's result is sharper. If we interpolate Krasikov's result with Lemma 3.12, we improve (3.39) a little bit.

**Lemma 3.14.** *Let $1 \leq c \leq 2$,*

$$\frac{n!}{(n+j)!} \max_{\lambda \geq 0} \left( \lambda^{j+c} e^{-\lambda} \left( L_n^j \right)^2 (\lambda) \right) \lesssim (1+n)^{(2-c)/6} (n+j+1)^{(3c-2)/2}, \quad j, n \in \mathbb{N}, \quad (3.41)$$

*or equivalently,*

$$\||w|^c V(h_j, h_k)(p,q)\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{b^{c/2}}(1+k)^{(2-c)/12}(j+1)^{(3c-2)/4}, \tag{3.42}$$

*where* $j, k \in \mathbb{N}$, $j \geq k$ *and* $w = p + iq \in \mathbb{C}$.

*Proof.* Two endpoint cases of (3.41) are $c = 1$ and $c = 2$.

The case $c = 2$ is given by taking $c = 2$ in (3.39).

The case $c = 1$ is almost in Theorem 3.13 except for $n = 0$. When $n = 0$, by Stirling formula,

$$\frac{1}{(j)!}\max_{\lambda\geq 0}\left(\lambda^{j+1}e^{-\lambda}\left(L_0^j\right)^2(\lambda)\right) = \frac{(j+1)^{j+1}e^{-(j+1)}}{j!} \lesssim \sqrt{j+1}.$$

Combining it with Theorem 3.13,

$$\frac{n!}{(n+j)!}\max_{\lambda\geq 0}\left(\lambda^{j+1}e^{-\lambda}\left(L_n^j\right)^2(\lambda)\right) \lesssim (1+n)^{1/6}\sqrt{n+j+1}, \quad j, n \in \mathbb{N}.$$

For any fixed $\lambda > 0$, vary the exponent $\alpha$ in $\lambda^{j+1+\alpha}e^{-\lambda}\left(L_n^j\right)^2(\lambda)$, where $0 \leq \mathcal{R}(\alpha) \leq 1$. Interpolating the two endpoint cases, (3.41) holds. □

*Remark* 3.15. Lemma 3.14 is stated for $1 \leq c \leq 2$. Because this is what we need in the present case. Nevertheless, using Krasikov's result, we can improve (3.39) for any $c \geq 1$.

*Remark* 3.16. The upper bound in Lemma 3.14 is not optimal. Consider two extreme cases $j = 0$ and $n = 0$ of

$$\frac{\sqrt{n!}}{\sqrt{(n+j)!}}\max_{\lambda\geq 0} e^{-\lambda/2}\lambda^{(c+j)/2}\left|L_n^j\right|(\lambda) \sim \||w|^c V(h_n, h_{n+j})(w)\|_{L^\infty}, \quad c \geq \frac{1}{2}. \tag{3.43}$$

[Sze75, Theorem 8.91.2, p. 241] says for any $a > 0$ and any fixed $j \in \mathbb{N}$,

$$\sup_{\lambda\geq a} e^{-\lambda/2}|\lambda|^{(c+j)/2}\left|L_n^j\right|(\lambda) \sim_j \langle n\rangle^{j/2+c/2-1/3}, \quad c \geq 1/2.$$

Taking $j = 0$, one can remove the constraint $\lambda \geq a > 0$ and show that $\max_{\lambda\geq 0} e^{-\lambda/2}|\lambda|^{c/2}|L_n|(\lambda) \sim \langle n\rangle^{c/2-1/3}$, $c \geq 1/2$. It gives a precise description of the asymptotic behavior of (3.43) for case $j = 0$.

For the case $n = 0$ of (3.43), by Stirling formula,

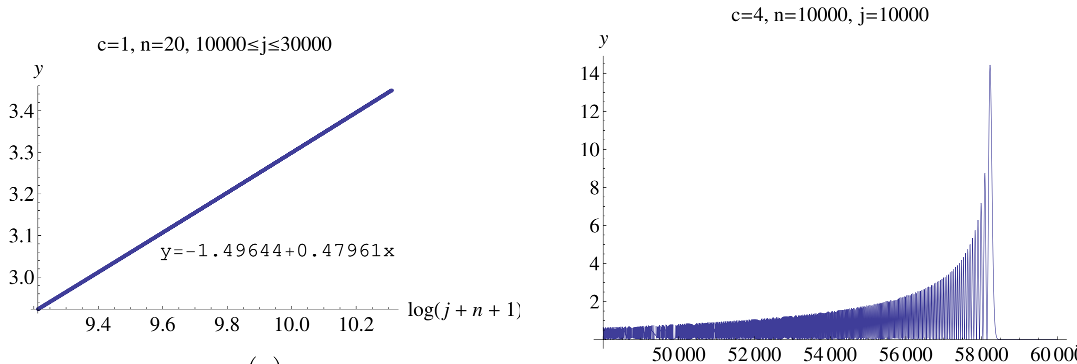
$$\frac{1}{\sqrt{j!}}\max_{\lambda\geq 0} e^{-\lambda/2}\lambda^{(c+j)/2}\left|L_0^j\right|(\lambda) = \frac{e^{-(c+j)/2}(c+j)^{(c+j)/2}}{\sqrt{j!}} \lesssim_c \langle j\rangle^{c/2-1/4}.$$

"Interpolating" the two cases, we conjecture

$$\||w|^c V(h_n, h_{n+j})(w)\|_{L^\infty} \lesssim_c \langle n\rangle^{-1/12}\langle n+j\rangle^{c/2-1/4}, \quad c \geq 1/2,\ j, n \in \mathbb{N}. \tag{3.44}$$

When $c = 1$, $n \geq 50$ and $j \geq 11$, by [KZ10, Theorem 2], (3.44) holds. For other cases, our numerical data, for example Figure 3.2, strongly suggests that (3.44) might hold.

Now we are ready to establish the collapsing estimate Theorem 2.6.




(a) (b)

In (A), to confirm the case $c = 1$, $y = \log\left(\frac{n!}{(n+j)!}\max_{\lambda\geq 0} e^{-\lambda}\lambda^{1+j}\left|L_0^j\right|^2(\lambda)\right)$ and $x = \log(j+n+1)$, where $n = 20$, the data almost lies on a line. For a larger range of $j$, the slope is close to 0.5. In (B), $y = \frac{n!(1+n)^{1/6}}{(n+j)!(n+j+1)^{c-1/2}} e^{-\lambda}\lambda^{(c+j)}\left|L_n^j\right|^2(\lambda)$. For fixed $c$, when we vary $n$ and $j$, from our numerical observation, $y$ is uniformly bounded. If we increase $c$, the bound increases.

Figure 3.2: Numerical Calculations

*Proof.* By the Parseval's theorem on $L^2([0,\pi/b])$,

$$
\begin{aligned}
&\left\||\nabla_x|^c\gamma(t,x,x)\right\|^2_{L_t^2L_x^2([0,\pi/b]\times\mathbb{R}^2)}\\
=&\left\||\nabla_x|^c\sum_{j,k\in\mathbb{N}} e^{-2b(j-k)it}\gamma_{jk}(x,x)\right\|^2_{L_t^2L_x^2([0,\pi/b]\times\mathbb{R}^2)}\\
=&\frac{\pi}{b}\sum_{m\in\mathbb{Z}}\left\||\nabla_x|^c\sum_{\substack{j-k=m\\ j,k\in\mathbb{N}}}\gamma_{jk}(x,x)\right\|^2_{L_x^2}.
\end{aligned}
\tag{3.45}
$$

We will express (3.45) by the Fourier transform of $|\nabla_x|^c\gamma_{jk}(x,x)$. Using the expression (3.36),

$$
\begin{aligned}
&\left(|\nabla_x|^c\widehat{\gamma_{jk}(x},x)\right)(\xi)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}^4} d\tilde{x}d\tilde{y}\,|\xi|^c\gamma_{jk}(\tilde{x},\tilde{y})\left(e^{-i\xi\tilde{x}}W(h_j)(\xi)\right)*\left(e^{-i\xi\tilde{y}}W(h_k)(\xi)\right)\\
&*\delta\left(\xi+\frac{b}{2}J(\tilde{x}-\tilde{y})\right),
\end{aligned}
$$

where $W(h_j)=W(h_j,h_j)$. To compute $(e^{-i\xi\tilde{x}}W(h_j)(\xi))*(e^{-i\xi\tilde{y}}W(h_k)(\xi))$, using tools from Appendix 3.6.3

$$
\begin{aligned}
&\left(e^{-i\xi\tilde{x}}W(h_j)(\xi)\right)*\left(e^{-i\xi\tilde{y}}W(h_k)(\xi)\right)\\
=&\int_{\mathbb{R}^2} d\tilde{\xi}\,e^{-i(\xi-\tilde{\xi})\tilde{x}}W(h_j)\left(\xi-\tilde{\xi}\right)e^{-i\tilde{\xi}\tilde{y}}W(h_k)\left(\tilde{\xi}\right)\\
=&\int_{\mathbb{R}^2} d\tilde{\xi}\,e^{-i(\xi-\tilde{\xi})\tilde{x}}W(h_j)\left(\tilde{\xi}-\xi\right)e^{-i\tilde{\xi}\tilde{y}}W(h_k)\left(\tilde{\xi}\right)\\
=&e^{-i\xi\tilde{x}/2}\int_{\mathbb{R}^2} d\tilde{\xi}\,W\left(\beta\left(\frac{\tilde{x}}{2}-\frac{J\xi}{b}\right)h_j,\beta\left(-\frac{\tilde{x}}{2}-\frac{J\xi}{b}\right)h_j\right)(\tilde{\xi})\overline{W\left(\beta\left(\frac{\tilde{y}}{2}\right)h_k,\beta\left(-\frac{\tilde{y}}{2}\right)h_k\right)}(\tilde{\xi})\\
=&\frac{2\pi e^{-i\xi\tilde{x}/2}}{b}\left\langle\beta\left(\frac{\tilde{x}}{2}-\frac{J\xi}{b}\right)h_j,\beta\left(\frac{\tilde{y}}{2}\right)h_k\right\rangle\left\langle\beta\left(-\frac{\tilde{y}}{2}\right)h_k,\beta\left(-\frac{\tilde{x}}{2}-\frac{J\xi}{b}\right)h_j\right\rangle\\
=&\frac{2\pi e^{-i\xi\tilde{x}/2}}{b}\left\langle\beta\left(-\frac{\tilde{y}}{2}\right)\beta\left(\frac{\tilde{x}}{2}-\frac{J\xi}{b}\right)h_j,h_k\right\rangle\left\langle\beta\left(\frac{\tilde{x}}{2}+\frac{J\xi}{b}\right)\beta\left(-\frac{\tilde{y}}{2}\right)h_k,h_j\right\rangle\\
=&\frac{2\pi e^{-i(\tilde{x}+\tilde{y})\xi}}{b}V\left(h_j,h_k\right)\left(\frac{\tilde{x}-\tilde{y}}{2}-\frac{J\xi}{b}\right)V\left(h_k,h_j\right)\left(\frac{\tilde{x}-\tilde{y}}{2}+\frac{J\xi}{b}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left(|\nabla_x|^c\widehat{\gamma_{jk}(x},x)\right)(\xi)\\
=&\frac{1}{b}\int_{\mathbb{R}^4} d\tilde{x}d\tilde{y}\,|\xi|^c\gamma_{jk}(\tilde{x},\tilde{y})\exp\left(-\frac{i}{2}\left[(\tilde{x}+\tilde{y})\xi+\frac{b}{2}\Omega(\tilde{x}+\tilde{y},\tilde{x}-\tilde{y})\right]\right)\times\\
&V(h_j,h_k)\left(\tilde{x}-\tilde{y}-\frac{J\xi}{b}\right)V(h_k,h_j)\left(\frac{J\xi}{b}\right).
\end{aligned}
$$

Next estimate (3.45), using the Fourier transform on $\tilde{x}+\tilde{y}$ and the Minkowski inequality,

$$
\begin{aligned}
(3.45) \lesssim\, & b^{-3} \sum_{m\in\mathbb{Z}} \left( \sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \left\| \int_{\mathbb{R}^2} d(\tilde{x}-\tilde{y})\, |\xi|^c \mathcal{F}_{\tilde{x}+\tilde{y}}(\gamma_{jk}) \left( \frac{1}{2}\left(\xi + \frac{bJ(\tilde{x}-\tilde{y})}{2}\right), \tilde{x}-\tilde{y} \right) \times \right.\right. \\
& \left.\left. V(h_j,h_k)\left(\tilde{x}-\tilde{y}-\frac{J\xi}{b}\right) V(h_k,h_j)\left(\frac{J\xi}{b}\right) \right\|_{L^2_\xi} \right)^2 \\
\lesssim\, & b^{-3} \sum_{m\in\mathbb{Z}} \left( \sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \|V(h_j,h_k)\|_{L^2} \|\gamma_{jk}\|_{L^2} \sup_{\xi\in\mathbb{R}^2} |\xi|^c |V(h_k,h_j)| \left(\frac{J\xi}{b}\right) \right)^2 \\
& \text{(Cauchy-Schwartz inequality)} \\
\lesssim\, & b^{-4} \sup_{m\in\mathbb{Z}} \left( \sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \frac{1}{\langle 2bj\rangle^s \langle 2bk\rangle^s} \sup_{\xi\in\mathbb{R}^2} |\xi|^{2c} |V(h_j,h_k)|^2 \left(\frac{J\xi}{b}\right) \right) \\
& \cdot \sum_{j,k\in\mathbb{N}} \langle 2bj\rangle^s \langle 2bk\rangle^s \|\gamma_{jk}\|_{L^2}^2. \\
& \left( \text{since } \|V(h_j,h_k)\|_{L^2}^2 = \frac{2\pi}{b} \langle h_k,h_k\rangle \langle h_j,h_j\rangle = \frac{2\pi}{b} \right)
\end{aligned}
$$

The estimate (2.29) reduces to show

$$
b^{-4} \sup_{m\in\mathbb{N}} \left( \sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \frac{1}{\langle 2bj\rangle^s \langle 2bk\rangle^s} \sup_{\xi\in\mathbb{R}^2} |\xi|^{2c} |V(h_j,h_k)|^2 \left(\frac{J\xi}{b}\right) \right) < \infty.
$$

Taking $c = 0$, by Lemma 3.12, for any $m \in \mathbb{N}$,

$$
\sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \frac{1}{\langle 2bj\rangle^s \langle 2bk\rangle^s} \left\| V(h_j,h_k)\left(\frac{J\xi}{b}\right) \right\|_{L^\infty}^2 \leq \sum_{\substack{j-k=m \\ k\in\mathbb{N}}} \frac{1}{\langle 2bj\rangle^s \langle 2bk\rangle^s} \leq \sum_{k\in\mathbb{N}} \frac{1}{\langle 2bk\rangle^{2s}},
$$

which is finite if $s > 1/2$. Taking $1 \leq c \leq 2$, by Lemma 3.14,

$$\sum_{\substack{j-k=m \\ j,k\in\mathbb{N}}} \frac{1}{\langle 2bj\rangle^s \langle 2bk\rangle^s} \left\| |\xi|^c V(h_j, h_k)\left(\frac{J\xi}{b}\right)\right\|_{L^\infty}^2 \lesssim \sum_{\substack{j-k=m \\ k\in\mathbb{N}}} \frac{b^c \langle k\rangle^{(2-c)/6} \langle j\rangle^{(3c-2)/2}}{\langle 2bj\rangle^s \langle 2bk\rangle^s}$$
$$\lesssim \frac{1}{b^{2s-c}} \sum_{k\in\mathbb{N}} \frac{1}{\langle k\rangle^{2s-4c/3+2/3}},$$

which is finite if $2s - 4c/3 + 2/3 > 1$. Setting $s = 1$, we get $1 \leq c < 5/4$.

Combining the low frequency case $c = 0$ and the high frequency case $1 \leq c < 5/4$ yields the estimate (2.29). □

## 3.4 Well-Posedness of the System

Before showing the local well-posedness result Theorem 2.3, we discuss Equation (2.12) in a case other than Equation (2.15) to demonstrate that $[\rho_Q * v, \phi]$ in Equation (2.15) is a trouble term. Equation (2.12) is well-posed in several spaces. The possible low regularity for the initial data when we can obtain a local well-posedness result is

$$\left\| H_{hx}^{1/8+\epsilon} H_{hy}^{1/8+\epsilon} \gamma_0(x,y) \right\|_{L^2_{x,y}} < \infty, \quad x, y \in \mathbb{R}^2, \quad \text{for arbitrary } \epsilon > 0, \tag{3.46}$$

where the norm is

$$\left\| H_h^{s/2} f \right\|_{L^2} = \left\| |\nabla|^s f \right\|_{L^2} + \left\| |x|^s f \right\|_{L^2}, \quad s \geq 0, \; f \in L^2(\mathbb{R}^2).$$

For the initial data (3.46), we acquire the following result.

**Theorem 3.17.** *Consider Equation (2.12) and suppose the initial condition $\gamma_0$ satisfies (3.46). Then Equation (2.12) has a mild solution for sufficiently short time $T$ in the Banach $\mathbf{N}_{HT}$, where the norm is defined as*

$$\begin{aligned}\|\gamma\|_{\mathbf{N}_{HT}} := &\left\| H_{hx}^{1/8+\epsilon} H_{hy}^{1/8+\epsilon}\gamma(t,x,y)\right\|_{L_t^\infty L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\ &+ \left\| |\nabla|^{1/2+2\epsilon}\rho_\gamma(t,x)\right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^2)},\end{aligned} \tag{3.47}$$

*where $\epsilon$ is the same in (3.46).*

*Remark* 3.18. Notice that the initial condition only requires that $\gamma_0$ is a Hilbert-Schmidt operator. It is not necessarily of trace class.

In order to use the technique in [GM17, Section 5, Section 6] [1] to prove Theorem 3.17, we need another version of the collapsing estimate

**Proposition 3.19.** *Suppose $\gamma(t,x,y) = e^{-i(H_x-\bar{H}_y)}\gamma_0(x,y)$ is the solution to the linear equation*

$$\begin{cases} i\,\partial_t\gamma(t) = [H,\gamma(t)], \\ \gamma(0,x,y) = \gamma_0(x,y) \in L_{x,y}^2, \end{cases} \tag{3.48}$$

*where $x,y\in\mathbb{R}^2$, the collapsing term $\rho_\gamma(t,x) = \gamma(t,x,x)$ satisfies*

$$\left\|\langle \tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\rho_\gamma(t,x)\right\|_{L_t^2 L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)} \lesssim_\epsilon \left\|\langle\nabla_x\rangle^{1/4+\epsilon}\langle\nabla_y\rangle^{1/4+\epsilon}\gamma_0(x,y)\right\|_{L_{x,y}^2}, \tag{3.49}$$

[1] The case studied in [GM17] is in three dimension. However we can modify the argument for our two dimensional problem Equation (2.12). Some steps in [GM17] need minor modification, yet the main idea is the same.

*where $\epsilon$ is any arbitrary small positive number.*

*Proof.* The operator $H$ is decomposed as (3.24) and $[H_h, H_r] = 0$. Since the rotation generated by the vector field $-iH_r$ satisfies

$$\left||\nabla|^s e^{-iH_r t} f\right|(x) = \left||\nabla|^s f\right|\left(e^{-iH_r t}x\right), \quad x \in \mathbb{R}^2$$

and $e^{-i(H_{rx}-\bar{H}_{ry})t}\gamma_0(x,y) = e^{-i(H_{rx}+H_{ry})t}\gamma_0(x,y)$,

$$\begin{aligned}
&\left\|\langle \tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\rho_\gamma(t,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}\\
=&\left\|\langle \tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\left(e^{-i\left(H_{rx}-\bar{H}_{ry}\right)t}e^{-i(H_{hx}-\bar{H}_{hy})t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}\\
=&\left\|\langle \tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\left(e^{-i(H_{hx}-\bar{H}_{hy})t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}.
\end{aligned}$$

Then the estimate (3.49) reduces to

$$\begin{aligned}
&\left\|\langle \tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\left(e^{-i(H_{hx}-\bar{H}_{hy})t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}\\
\lesssim_\epsilon&\left\|\langle\nabla_x\rangle^{1/4+\epsilon}\langle\nabla_y\rangle^{1/4+\epsilon}\gamma_0(x,y)\right\|_{L^2_{x,y}},
\end{aligned}$$

where the collapsing term corresponds to the equation $i\,\partial_t\gamma = [H_h,\gamma]$. By the Lens transform [Tao09]

$$Ł(u)(t,x) \coloneqq \frac{1}{\cos bt}u\left(\frac{\tan bt}{b}, \frac{x}{\cos bt}\right)e^{-\left(ib|x|^2\tan bt\right)/4}, \quad t\in\mathbb{R},\ x\in\mathbb{R}^2, \tag{3.50}$$

which maps the solution $u(t,x)$ of $i\,\partial_t u = -\Delta u$ to the solution of $i\,\partial_t Ł(u) = H_h Ł(u)$, we obtain the identity

$$\begin{aligned}&\left\|\langle\tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\left(e^{-i(H_{hx}-\bar{H}_{hy})t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}\\ =&\left\||\nabla_x|^{1/2+2\epsilon}\left(e^{i(\Delta_x-\Delta_y)t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2}.\end{aligned}$$

Finally, the estimate (3.49) reduces to the Laplacian case

$$\left\||\nabla_x|^{1/2+2\epsilon}\left(e^{i(\Delta_x-\Delta_y)t}\gamma_0\right)(x,x)\right\|_{L_t^2L_x^2}\lesssim_\epsilon\left\|\langle\nabla_x\rangle^{1/4+\epsilon}\langle\nabla_y\rangle^{1/4+\epsilon}\gamma_0(x,y)\right\|_{L_{x,y}^2},$$

which is proved in [GM17, CHP17]. $\square$

Since the Hermite operator $H_h$ dominates $-\Delta+1$ in the sense $\|\langle-\Delta\rangle^{s/2}f\|_{L^2}\lesssim\left\|H_h^{s/2}f\right\|_{L^2}$ for $s\geq 0$, as a corollary of Proposition 3.19

$$\left\|\langle\tan bt\rangle^{-1/2-\epsilon}|\nabla_x|^{1/2+2\epsilon}\rho_\gamma(t,x)\right\|_{L_t^2L_x^2([-\pi/2b,\pi/2b]\times\mathbb{R}^2)}\lesssim_\epsilon\left\|H_{hx}^{1/8+\epsilon}H_{hy}^{1/8+\epsilon}\gamma_0(x,y)\right\|_{L_{x,y}^2}. \tag{3.51}$$

using this estimate (3.51) and the scheme in [GM17], Theorem 3.17 follows.

When it comes to Equation (2.15), if we expect to establish a local well-posedness result when

$$\left\|H_{hx}^{s/2}H_{hy}^{s/2}Q_0(x,y)\right\|_{L_{x,y}^2}<\infty,$$

we need to deal with terms, for example $\left\||\nabla_x|^s(\rho_Q*v)H_{hy}^{s/2}\bar{\Pi}_\phi(x,y)\right\|_{L_{x,y}^2}$. However $H_{hy}^{s/2}\bar{\Pi}_\phi$ is not translation invariant. After integrating over $y$, we are faced with $\||x|^s|\nabla_x|^s(\rho_Q*v)\|_{L^2}$. For the linear equation $i\,\partial_tQ=[H+\rho_Q*v,Q]$ and $Q(t=0)=$

$Q_0$, $\||x|^s |\nabla_x|^s (\rho_Q * v)\|_{L^2_{I_T} L^2}$ is controlled by $\left\| H_{hx}^{s/2} H_{hy}^{s/2} Q_0(x,y) \right\|_{tr}$. But it may not be controlled by $\left\| H_{hx}^{s/2} H_{hy}^{s/2} Q_0(x,y) \right\|_{L^2_{x,y}}$. Therefore we can not close the argument to obtain a local well-posedness result of Equation (2.15). That is why we stick to the structure of Equation (2.15) and use norms arising from $H$, i.e. Definition 2.1. The operator $H$ is more compatible with the stationary solution $\bar{\Pi}_\phi$ than $H_h$. Hence we can deal with $\langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} [\rho_Q * v, \bar{\Pi}_\phi]$.

Next we prove Theorem 2.3 the local wellposedness result of Equation (2.15).

*Proof.* By Duhamel's formulation, we define the solution map $\Phi$ and the solution ball $sol_T$ for the contraction mapping principle,

$$\Phi(Q)(t,x,y) := e^{-i(H_x - \bar{H}_y)t} Q_0 - i \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} [v * \rho_Q, Q + \bar{\Pi}_\phi](\tau)\, d\tau, \quad (3.52)$$

$$sol_T := \left\{ Q(t,x,y) \middle| \|Q(t,x,y)\|_{\mathbf{N}_T^H} \le C \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q_0(x,y) \right\|_{L^2_{x,y}} \right\}, \quad (3.53)$$

where parameters $T$ and $C > 1$ are to be determined later.

1. Show $\Phi$ maps $sol_T$ to itself.

Suppose $Q \in sol_T$. By Theorem 2.6 and Proposition 3.11,

$$\|e^{-i(H_x - \bar{H}_y)t} Q_0\|_{\mathbf{N}_T^H} \lesssim_T \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q_0 \right\|_{L^2_{x,y}}.$$

Choosing $T = \pi/4b$, then $C$ is the constant such that

$$\|e^{-i(H_x - \bar{H}_y)t} Q_0\|_{\mathbf{N}_T^H} \le C \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q_0 \right\|_{L^2_{x,y}} /2.$$

For the nonlinear part, claim the estimate

$$\begin{aligned}&\left\| \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} [v * \rho_Q, Q + \bar{\Pi}_\phi](\tau)\, d\tau \right\|_{\mathbf{N}_T^H} \\ &\lesssim \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} [v * \rho_Q, Q + \bar{\Pi}_\phi] \right\|_{L^1_{I_T} L^2_{x,y}} . \end{aligned} \tag{3.54}$$

The proof of (3.54) is twofold. On one hand, to control the Strichartz norm,

$$\left\| \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} \underbrace{\langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} [v * \rho_Q, Q + \bar{\Pi}_\phi](\tau)}_{F_1(\tau,x,y)} \, d\tau \right\|_{L^q_{I_T} L^r_x L^2_y} ,$$

suppose $G(t,x,y)$ is in the dual Strichartz space $L^{q'}_{I_T} L^{r'}_x L^2_y$, where

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Using the dual characterization of $L^p$ spaces

$$\begin{aligned}&\int_{I_T} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dt dx dy\, \bar{G}(t,x,y) \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} F_1(\tau,x,y) d\tau \\ &= \int_{I_T} \int_{\mathbb{R}^2 \times \mathbb{R}^2} d\tau dx dy\, F_1(\tau,x,y) \int_\tau^T \overline{e^{-i(H_x - \bar{H}_y)(\tau - t)} G}(t,x,y)\, dt \\ &\le \int_{I_T} d\tau\, \|F_1(\tau,x,y)\|_{L^2_x L^2_y} \left\| \int_\tau^T \overline{e^{-i(H_x - \bar{H}_y)(\tau - t)} G}(t,x,y)\, dt \right\|_{L^\infty_{I_T} L^2_{x,y}} \\ &\lesssim \int_{I_T} d\tau\, \|F_1(\tau,x,y)\|_{L^2_x L^2_y} \|G(t,x,y)\|_{L^{q'}_{I_T} L^{r'}_x L^2_y} ,\end{aligned}$$

(by Proposition 3.11 the dual Strichartz estimate (3.34)) we obtain,

$$\left\| \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} F_1(\tau, x, y)\, d\tau \right\|_{L^q_{I_T} L^r_x L^2_y} \lesssim \| F_1(t, x, y) \|_{L^1_t L^2_{x,y}([0,T] \times \mathbb{R}^4)} .$$

The argument for the norm $L^q_{I_T} L^r_y L^2_x$ is the same. On the other hand, to control the collapsing term

$$\left\| \langle \nabla_x \rangle^{9/8} \left( \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} \underbrace{[v * \rho_Q, Q + \bar{\Pi}_\phi](\tau)}_{F_2(\tau,x,y)}\, d\tau \right)(t, x, x) \right\|_{L^2_{I_T} L^2_x},$$

applying Theorem 2.6 and the Minkowski inequality,

$$\begin{aligned}
&\left\| \langle \nabla_x \rangle^{9/8} \left( \int_0^t e^{-i(H_x - \bar{H}_y)(t-\tau)} F_2(\tau, x, y)\, d\tau \right)(t, x, x) \right\|_{L^2_{I_T} L^2_x} \\
\leq &\left\| \int_0^T \left\| \langle \nabla_x \rangle^{9/8} \left( e^{-i(H_x - \bar{H}_y)(t-\tau)} F_2(\tau, x, y) \right)(t, x, x) \right\|_{L^2_x} d\tau \right\|_{L^2_{I_T}} \\
\leq &\int_0^T d\tau \left\| \langle \nabla_x \rangle^{9/8} \left( e^{-i(H_x - \bar{H}_y)(t-\tau)} F_2(\tau, x, y) \right)(t, x, x) \right\|_{L^2_{I_T} L^2_x} \\
\lesssim &\int_0^T d\tau \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} F_2(\tau, x, y) \right\|_{L^2_{x,y}} \quad \text{(by Theorem 2.6)}.
\end{aligned}$$

According to the estimate (3.54), the problem is reduced to estimate quantities

1. $\left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} [v * \rho_Q, Q] \right\|_{L^1_t L^2_{x,y}([0,T] \times \mathbb{R}^4)}$,
2. $\left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} [v * \rho_Q, \bar{\Pi}_\phi] \right\|_{L^1_t L^2_{x,y}([0,T] \times \mathbb{R}^4)}$.

Since the commutation relation does not play a role of our analysis, it suffices to prove one of the two terms in the commutation relation. The other one is handled similarly.

Considering $\left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} \left( \left( v * \rho_Q \right)(t,x) Q(t,x,y) \right) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)}$, based on the observation (3.29), we have

$$
\begin{aligned}
&\lesssim \left\| D_x \langle \bar{H}_y \rangle^{1/2} \left( v * \rho_Q \right)(t,x) Q(t,x,y) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\
&\quad + \left\| \langle \bar{H}_y \rangle^{1/2} \left( v * \rho_Q \right)(t,x) Q(t,x,y) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\
&\lesssim \left\| 2\partial_{z_x} \left( v * \rho_Q \right)(t,x) \langle \bar{H}_y \rangle^{1/2} Q(t,x,y) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\
&\quad + \left\| \left( v * \rho_Q \right)(t,x) \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q(t,x,y) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)},
\end{aligned}
$$

and by the virtue of Hölder inequality, Sobolev inequality, Lemma 3.10 and Young's convolution inequality, we obtain

$$
\begin{aligned}
&\left\| 2\partial_{z_x} \left( v * \rho_Q \right)(t,x) \langle \bar{H}_y \rangle^{1/2} Q(t,x,y) \right\|_{L_t^1 L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\
\lesssim & T^{1/2} \left\| |\nabla_x| (v * \rho_Q)(t,x) \right\|_{L_t^2 L_x^{\frac{16}{7}}([0,T]\times\mathbb{R}^2)} \left\| \langle \bar{H}_y \rangle^{1/2} Q(t,x,y) \right\|_{L_t^\infty L_x^{16} L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \\
\lesssim & T^{1/2} \left\| |\nabla_x|^{9/8} \rho_Q(t,x) \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^2)} \left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q(t,x,y) \right\|_{L_t^\infty L_{x,y}^2([0,T]\times\mathbb{R}^4)} \\
\lesssim & T^{1/2} \left\| Q(t) \right\|_{\mathbf{N}_T^H}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|(v*\rho_Q)(t,x)\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q(t,x,y)\right\|_{L_t^1L_{x,y}^2([0,T]\times\mathbb{R}^4)}\\
\lesssim&T^{1/4}\left\|(v*\rho_Q)(t,x)\right\|_{L_t^2L_x^4([0,T]\times\mathbb{R}^2)}\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q(t,x,y)\right\|_{L_t^4L_x^4L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}\\
\lesssim&T^{1/4}\left\||\nabla_x|^{1/2}\rho_Q(t,x)\right\|_{L_t^2L_x^2([0,T]\times\mathbb{R}^2)}\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q(t,x,y)\right\|_{L_t^4L_x^4L_y^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}\\
\lesssim&T^{1/4}\left\|Q(t)\right\|_{\mathbf{N}_T^H}^2.
\end{aligned}
$$

Next we consider $\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}[v*\rho_Q,\bar{\Pi}_\phi]\right\|_{L_t^1L_{x,y}^2([0,T]\times\mathbb{R}^4)}$, by direct computation

$$
\begin{aligned}
\bar{D}_y\bar{\Pi}_\phi(x,y)&=\left(2\partial_z\phi(x-y)+\frac{b}{2}(\bar{z}_x-\bar{z}_y)\phi(x-y)\right)e^{-ib\Omega(x,y)/2},\\
D_x\bar{\Pi}_\phi(x,y)&=\left(-2\partial_{\bar{z}}\phi(x-y)-\frac{b}{2}(z_x-z_y)\phi(x-y)\right)e^{-ib\Omega(x,y)/2},\\
D_x\bar{D}_y\bar{\Pi}_\phi(x,y)&=(-4\partial_{\bar{z}}\partial_z\phi(x-y)-b(z_x-z_y)\partial_z\phi(x-y)\\
&\qquad-b(\bar{z}_x-\bar{z}_y)\partial_{\bar{z}}\phi(x-y)-\frac{b^2}{4}|x-y|^2\phi(x-y)\Big)e^{-ib\Omega(x,y)/2},
\end{aligned}
$$

integrating over $x$ or $y$, we obtain

$$
\begin{aligned}
\left\|\bar{D}_y\bar{\Pi}_\phi(x,y)\right\|_{L^2_{x(y)}}&=\left\|\bar{D}\phi\right\|_{L^2}\lesssim\left\|\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}\\
\left\|D_x\bar{\Pi}_\phi(x,y)\right\|_{L^2_{x(y)}}&=\left\|D\phi\right\|_{L^2}\lesssim\left\|\langle H\rangle^{1/2}\phi\right\|_{L^2},\\
\left\|D_x\bar{D}_y\bar{\Pi}_\phi(x,y)\right\|_{L^2_{x(y)}}&=\left\|D\bar{D}\phi\right\|_{L^2}\lesssim\left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}.
\end{aligned}
$$

Combining the above estimates,

$$\begin{aligned}
&\left\|\langle H_x\rangle^{1/2}\langle \bar{H}_y\rangle^{1/2}[v*\rho_Q,\bar{\Pi}_\phi]\right\|_{L^1_t L^2_{x,y}([0,T]\times\mathbb{R}^4)}\\
\lesssim&\left\|\langle\nabla_x\rangle\rho_Q(t,x)\right\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^2)}\left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}\\
\lesssim&T^{1/2}\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q_0(x,y)\right\|_{L^2_{x,y}}\left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}
\end{aligned}$$

If necessary, shrink the interval $I_T$ such that

$$\left\|\int_0^t e^{-i(H_x-\bar{H}_y)(t-\tau)}[v*\rho_Q,Q+\bar{\Pi}_\phi](\tau)\,d\tau\right\|_{\mathbf{N}_T^H}\le\frac{C}{2}\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q_0(x,y)\right\|_{L^2_{x,y}}.$$

Thus $\Phi$ maps $sol_T$ to itself.

2. Show $\Phi$ is a contraction map.

For any $Q_1$, $Q_2\in sol_T$, similarly as step 1,

$$\begin{aligned}
&\left\|\Phi(Q_1)-\Phi(Q_2)\right\|_{\mathbf{N}_T^H}\\
\le&\left\|\int_0^t d\tau\,e^{-i(H_x-\bar{H}_y)(t-\tau)}[v*\rho_{Q_1}-v*\rho_{Q_2},\bar{\Pi}_\phi]\right\|_{\mathbf{N}_T^H}\\
&+\left\|\int_0^t d\tau\,e^{-i(H_x-\bar{H}_y)(t-\tau)}[v*\rho_{Q_1}-v*\rho_{Q_2},Q_1]\right\|_{\mathbf{N}_T^H}\\
&+\left\|\int_0^t d\tau\,e^{-i(H_x-\bar{H}_y)(t-\tau)}[v*\rho_{Q_2},Q_1-Q_2]\right\|_{\mathbf{N}_T^H}\\
\lesssim&T^{1/2}\left\|\langle\nabla_x\rangle(\rho_{Q_1}-\rho_{Q_2})\right\|_{L^2_{I_T}L^2_x}\left\|\langle H\rangle^{1/2}\langle\bar{H}\rangle^{1/2}\phi\right\|_{L^2}\\
&+\max\{T^{1/2},T^{1/4}\}\left\|Q_1-Q_2\right\|_{\mathbf{N}_T^H}\left(\|Q_1\|_{\mathbf{N}_T^H}+\|Q_2\|_{\mathbf{N}_T^H}\right)\\
\lesssim&\max\{T^{1/2},T^{1/4}\}\left\|Q_1-Q_2\right\|_{\mathbf{N}_T^H}\left\|\langle H_x\rangle^{1/2}\langle\bar{H}_y\rangle^{1/2}Q_0(x,y)\right\|_{L^2_{x,y}}.
\end{aligned}$$

If needed, choose a smaller $T$ such that $\|\Phi(Q_1)-\Phi(Q_2)\|_{\mathbf{N}_T^H} \leq \|Q_1-Q_2\|_{\mathbf{N}_T^H}/2$.

Then by the contraction mapping principle, $\Phi$ has a fixed point in $sol_T$, i.e. Equation (2.15) is locally well-posed. $\square$

*Remark* 3.20. There are two families of stationary solutions $\Pi_\phi$ and $\bar{\Pi}_\phi$ (see Section 3.6.2). The reason for only $\bar{\Pi}_\phi$ is used in our pertubation problem is twofold. On one hand, $\bar{\Pi}_\phi$ recovers the Fermi-Dirac distribution. On the other hand, suppose we use the stationary solution $\Pi_\phi$ instead of $\bar{\Pi}_\phi$. By the product rule of the covariant derivative $D$, $D(fg) = (Df)g - 2f\partial_{\bar{z}}g$,

$$\begin{aligned}
&D_x\bar{D}_y\left(\rho_u * v(x)\Pi_\phi(x,y)\right)\\
=&D_x\left(\rho_u * v\right)(x)\bar{D}_y\Pi_\phi(x,y) + \left(\rho_u * v\right)(x)\left(-2\partial_{\bar{z}_x}\right)\bar{D}_y\Pi_\phi(x,y)\\
\text{or } &D_x\bar{D}_y\left(\rho_u * v(x)\Pi_\phi(x,y)\right)\\
=&\left(-2\partial_{\bar{z}_x}\right)\left(\rho_u * v\right)(x)\bar{D}_y\Pi_\phi(x,y) + \left(\rho_u * v\right)(x)D_x\bar{D}_y\Pi_\phi(x,y).
\end{aligned} \tag{3.55}$$

Since we do not have an estimate for $D_x\rho_u(t,x)$, we use the form (3.55) to continue our argument. A direct computation shows

$$\bar{D}_y\Pi_\phi(x,y) = \left(2\partial_z\phi(x-y) - \frac{b}{2}(\bar{z}_x+\bar{z}_y)\phi(x-y)\right)e^{ib\Omega(x,y)/2}.$$

$\left|\bar{D}_y\Pi_\phi(x,y)\right|$ is not translation invariant. Therefore in order to estimate

$$\left\|-2\partial_{\bar{z}_x}(\rho_u * v)(t,x)\bar{D}_y\Pi_\phi(x,y)\right\|_{L^2_{x,y}},$$

we need to control $\| x |\nabla_x| \rho_u(t,x) \|_{L^2_x}$, which is not possible by using $\mathbf{N}_T^H$.

## 3.5 Conclusion

In this chapter, we obtained a local well-posed result of Equation (2.15) and a new collapsing estimate Theorem 2.6. However the estimate is not sharp since we do not have an optimal control of associated Laguerre polynomials (see Remark 3.16).

The ultimate goal of Theorem 2.3 is to acquire a low regularity result, for example a local well-posedness result for the initial data

$$\left\| \langle H_x \rangle^{s/2} \langle \bar{H}_y \rangle^{s/2} Q_0(x,y) \right\|_{L^2_{x,y}} < \infty, \quad s < 1,$$

According to Remark 3.16 and the proof of Theorem 2.6, we have a little gain of derivatives for the collapsing term when $s > 1/3$. We conjecture that the best case might be $s = 1/3 + \epsilon$. However it requires a fractional Leibniz rule for $\langle H \rangle^{s/2}(fg)$, which currently is beyond our ability.

Another direction is to establish a global well-posedness result when

$$\left\| \langle H_x \rangle^{1/2} \langle \bar{H}_y \rangle^{1/2} Q_0(x,y) \right\|_{Tr} < \infty.$$

A formal computation shows that the total energy (3.56) of Equation (2.15) is conserved

$$\mathcal{E}(Q) = Tr\left( H^{1/2} Q H^{1/2} \right) + \frac{1}{2} \int_{\mathbb{R}^2} \left( v * \rho_Q \right)(x) \rho_Q(x)\, dx, \tag{3.56}$$

which brings hope for the global well-posedness result at the energy level. In order to establish the global well-posedness result, we need to control the trace norm of the forcing term $\rho_Q \bar{\Pi}_\phi$, which is a composition of multiplication operator $\rho_Q$ and a non-compact operator $\bar{\Pi}_\phi$. However, the operator $H$ is no longer the Laplace operator, so the Birman-Solomjak inequality [Sim05, Theorem 4.5], for $1 \le p \le 2$,

$$\|f(x)g(-i\nabla)\|_{\mathcal{L}^p} \lesssim_p \|f\|_{l^p L^2} \|g\|_{l^p L^2}, \tag{3.57}$$

where $\mathcal{L}^p$ is the $p$-th Schatten norm, can not be applied. Besides, due to the special spectral structure of $H$, it is challenging to develop a corresponding version of this inequality (3.57) for $H$, which, to the author's knowledge, is not available in the existing literature. The author is working on obtaining the essential estimates.

## 3.6 Appendix

### 3.6.1 Heisenberg Group

[Fol89, Chapter 1]Let us review the Heisenberg group $H_1$ with the group law

$$(p_1, q_1, t_1) \cdot (p_2, q_2, t_2) = \left(p_1 + p_2, q_1 + q_2, t_1 + t_2 + b\frac{\Omega((p_1, q_1), (p_2, q_2))}{2}\right),$$

where $p_i, q_i \in \mathbb{R}$, $t_i \in \mathbb{R}$, and impose a complex structure on $\mathbb{R}^2$, $z = p + q\,i$.

Identify the tangent space $TH_1$ with $\mathbb{R}^3 \times \mathbb{R}^3$ and its basis by $\{\partial_p, \partial_q, \partial_t\}$. Then the differential of the left multiplication $L_g$, where $g = (p_g, q_g, t_g)$, is

$$DL_g(\partial_p, \partial_q, \partial_t) = (\partial_p, \partial_q, \partial_t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -bq_g/2 & bp_g/2 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{h}_1$ consisting of left invariant vector fields is

$$\mathfrak{h}_1 = \mathbb{R}\text{-span}\left\{\partial_p - b\frac{q}{2}\partial_t,\ \partial_q + b\frac{p}{2}\partial_t,\ \partial_t\right\},$$

and the corresponding complexified space is

$$\mathfrak{h}_1^{\mathbb{C}} = \mathbb{C}\text{-span}\left\{2\partial_{\bar{z}} + i\frac{bz}{2}\partial_t,\ 2\partial_z - i\frac{b\bar{z}}{2}\partial_t,\ \partial_t\right\}.$$

We will think of $D$ and $D^*$ as vector fields of $\mathfrak{h}_1^{\mathbb{C}}$ in the following way. Denote

$$D_{H_1} = -2\partial_{\bar{z}} - i\frac{bz}{2}\partial_t, \quad D_{H_1}^* = 2\partial_z - i\frac{b\bar{z}}{2}\partial_t.$$

Suppose $\tilde{f} \in \mathcal{S}(H_1)$ and apply the inverse Fourier transform on $t$ variable,

$$\check{D}_{H_1}\check{\tilde{f}} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(-2\partial_{\bar{z}} - \frac{bz\tau}{2}\right)\tilde{f}(q,p,t)e^{it\tau}\,dt.$$

On the piece $\tau = 1$, $D_{H_1}$ and $D_{H_1}^*$ correspond to $D$ and $D^*$ respectively.

To make this correspondence rigorous, consider a quotient group $H_1^{red}$ of $H_1$

$$H_1^{red} := H_1 / \{(0,0,t) | t \in 2\pi\mathbb{Z}\}, \ \{(0,0,t) | t \in 2\pi\mathbb{Z}\} \subset C(H_1).$$

For a $f$ on $\mathbb{R}^2$, it is lifted to $H_1$ by defining

$$\tilde{f}(p,q,t) := \sqrt{2\pi}\exp(-ti) f(p,q). \tag{3.58}$$

Through the definition (3.58), the correspondence between $D(D^*)$ and $D_{H_1}\left(D^*_{H_1}\right)$ is

$$D\tilde{f}(p,q,t) = D^*_{H_1}\tilde{f}(p,q,t), \quad D^*\tilde{f}(p,q,t) = D^*_{H_1}\tilde{f}(p,q,t). \tag{3.59}$$

We can also relate the twisted convolution defined in (3.8) to the group convolution on $H_1$,

$$\begin{aligned}
\left(\tilde{f} * \tilde{g}\right)(p,q,t) &= \int_{H_1^{red}} \tilde{f}\left((p,q,t)\cdot(\tilde{p},\tilde{q},\tilde{t})^{-1}\right)\tilde{g}(\tilde{p},\tilde{q},\tilde{t})\, d\tilde{p}d\tilde{q}d\tilde{t} \\
&= \int_{H_1^{red}} \tilde{f}\left(p-\tilde{p}, q-\tilde{q}, t-\tilde{t}-b\frac{\Omega\left((p,q),(\tilde{p},\tilde{q})\right)}{2}\right)\tilde{g}(\tilde{p},\tilde{q},\tilde{t})\, d\tilde{p}d\tilde{q}d\tilde{t} \\
&= 2\pi\exp(-ti)\int_{\mathbb{R}^2} f(p-\tilde{p}, q-\tilde{q}) g(\tilde{p},\tilde{q}) \exp\left(ib\frac{\Omega\left((p,q),(\tilde{p},\tilde{q})\right)}{2}\right) d\tilde{p}d\tilde{q} \\
&= 2\pi\exp(-ti)\left(f \natural g\right)(p,q),
\end{aligned}$$

i.e. $\tilde{f} * \tilde{g} = \sqrt{2\pi}\widetilde{f \natural g}$.

**Lemma 3.21.** *Let $G$ be a Lie group endowed with a left invariant Haar measure*

*$d\mu$, then*

$$L_X\left(f * g\right) = f * L_X g, \quad X \in \mathfrak{g} \tag{3.60}$$

*where $L_X$ denotes the Lie derivative by $X$ and $*$ denotes the convolution on $G$*

$$\left(f * g\right)\left(x\right) := \int_G f(xy^{-1})g(y)dy, \quad x \in G.$$

*Furthermore, (3.60) holds for the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$.*

*Proof.* Suppose $X \in \mathfrak{g}$, let $\exp(tX)$ denote the one parameter subgroup generated by $X$ and $\exp(tX).x$ denote the action of $\exp(tX)$ on $G$, i.e. $x \in G$ travels along the flow generated by $X$. Then

$$\begin{aligned}
\int_G f\left((\exp(tX).x)y^{-1}\right) g(y)\, dy &= \int_G f\left(x \exp(tX) y^{-1}\right) g(y)\, dy \\
&= \int_G f\left(x \left(y \exp(-tX)\right)^{-1}\right) g(y)\, dy \\
&= \int_G f\left(xy^{-1}\right) g\left(\exp(tX).y\right)\ L^*_{\exp(tX)} dy \\
&= \int_G f\left(xy^{-1}\right) g\left(\exp(tX).y\right)\ dy
\end{aligned}$$

which implies the identity (3.60). □

### 3.6.2 Stationary Solutions

We use relations (3.59) to find two families of stationary solutions to Equation (2.12).

**Proposition 3.22.** *Suppose $v \in L^1\left(\mathbb{R}^2\right)$, there are two families of stationary solu-*

*tions to Equation (2.12),*

*(i)* $\Pi_\phi(x,y) = \phi(x-y)\exp\left(ib\dfrac{\Omega(x,y)}{2}\right)$*, for arbitrary* $\phi$ *on* $\mathbb{R}^2$*,*

*(ii)* $\bar{\Pi}_\phi(x,y) = \phi(x-y)\exp\left(-ib\dfrac{\Omega(x,y)}{2}\right)$*, where* $\phi$ *is of radial symmetry, i.e.* $\phi(x) = \phi(|x|)$*.*

*Proof.* By the correspondence (3.59), we regard $D$ and $D^*$ as vector fields of $H_1$. Since the Lebesgue measure on $H_1^{red}$ is bi-invariant and the group convolution on $H_1$ is related to the twisted convolution by $\tilde{f} * \tilde{g} = \sqrt{2\pi}\widetilde{f \natural g}$, using Lemma 3.21, we conclude that the Hamiltonian $H = D^*D$ commutes with the twisted convolution $\natural$. As a result,

$$\begin{aligned} &[D^*D, \Pi_\phi] = D^*[D,\Pi_\phi] + [D^*,\Pi_\phi]D = 0 \\ \Longrightarrow\ &H_x \int_{\mathbb{R}^2} \Pi_\phi(x,y) f(y)dy - \int_{\mathbb{R}^2} \Pi_\phi(x,y) H_y f(y)dy \\ &= \int_{\mathbb{R}^2} \left(H_x \Pi_\phi(x,y) - \bar{H}_y \Pi_\phi(x,y)\right) f(y)dy = 0,\ \forall f \in \mathcal{S}(\mathbb{R}^2) \end{aligned}$$

Besides $\Pi_\phi(x,x) = \phi(0)$ and $v*\phi(0) = \phi(0)\int v(x)\,dx$ are constant, $\Pi_\phi$ is a stationary solution to (2.12).

Meanwhile, if we calculate $\left(H_x - \bar{H}_y\right)\bar{\Pi}_\phi$ directly,

$$\begin{aligned} \left(H_x - \bar{H}_y\right)\bar{\Pi}_\phi &= \left(H_x - \bar{H}_x - \bar{H}_y + H_y\right)\bar{\Pi}_\phi + \left(\bar{H}_x - H_y\right)\bar{\Pi}_\phi \\ &= 2ib\left(xJ\nabla_x + yJ\nabla_y\right)\bar{\Pi}_\phi \\ &= 2ib(x-y)^T J\left(\nabla_x \bar{\phi}(x-y)\right)\exp\left(-\frac{ib\Omega(x,y)}{2}\right), \end{aligned}$$

which vanishes if $\phi$ is a function of radial symmetry. $\square$

### 3.6.3 Transform

We list some important results about the Fourier-Wigner transform $V$ and the Wigner transform $W$ from [Fol89, Chapter 1]. In the paper, we choose the reduced Planck constant $\hbar$ in [Fol89, Chapter 1] to be $b$ and use the following results when the dimension $d = 1$.

**Proposition 3.23.** *[Fol89, Proposition 1.42]*

$$\langle V(f_1, g_1), V(f_2, g_2)\rangle = \left(\frac{2\pi}{b}\right)^d \langle f_1, f_2\rangle \langle g_2, g_1\rangle, \quad f_j, g_j \in L^2(\mathbb{R}^d),\ j = 1, 2.$$

**Proposition 3.24.** *[Fol89, Proposition 1.47] Suppose $f_j, g_j \in L^2(\mathbb{R}^d)$,*

$$\overline{V(f_1, g_1)} \natural \overline{V(f_2, g_2)} = \left(\frac{2\pi}{b}\right)^d \langle g_2, f_1\rangle \overline{V(f_2, g_1)}.$$

**Proposition 3.25.** *[Fol89, Proposition 1.94]*

$$\begin{aligned} W\left(\beta(a,e)f, \beta(c,d)g\right)(\xi, x) = \exp&\left(-\frac{ib}{2}\Omega\left((a,e),(c,d)\right) + i\left\langle (a,e)-(c,d), (\xi,x)\right\rangle\right) \\ &\cdot W(f,g)\left(\xi - \frac{b(e+d)}{2}, x + \frac{b(a+c)}{2}\right). \end{aligned}$$

*where $a, e, c, d, x, \xi \in \mathbb{R}^d$.*

Hermite functions and associated Laguerre polynomials are related by the following two theorems.

**Theorem 3.26.** *[Fol89, Theorem 1.104] Suppose $p, q \in \mathbb{R}$, and $w = p + iq$. Then*

$$V(h_j, h_k)(p,q) = \begin{cases} \sqrt{\frac{k!}{j!}}\left(\sqrt{\frac{b}{2}}w\right)^{j-k} e^{-b|w|^2/4} L_k^{j-k}\left(\frac{b|w|^2}{2}\right), & j \geq k \\ (-1)^{j+k}\sqrt{\frac{j!}{k!}}\left(\sqrt{\frac{b}{2}}\bar{w}\right)^{k-j} e^{-b|w|^2/4} L_j^{k-j}\left(\frac{b|w|^2}{2}\right), & j \leq k \end{cases}$$

**Theorem 3.27.** *[Fol89, Theorem 1.105] Suppose $x, \xi \in \mathbb{R}$ and $z = x + i\xi$. Then*

$$W(h_j, h_k)(\xi, x) = \begin{cases} (-1)^k \frac{2}{b}\sqrt{\frac{k!}{j!}}\left(\sqrt{\frac{2}{b}}\bar{z}\right)^{j-k} L_k^{j-k}\left(\frac{2|z|^2}{b}\right) e^{-|z|^2/b}, & j \geq k \\ (-1)^j \frac{2}{b}\sqrt{\frac{j!}{k!}}\left(\sqrt{\frac{2}{b}}z\right)^{k-j} L_j^{k-j}\left(\frac{2|z|^2}{b}\right) e^{-|z|^2/b}, & j \leq k \end{cases}$$

Let $\mu$ be the Metaplectic representation from $Mp(2d, \mathbb{R})$ to $U\left(L^2(\mathbb{R}^d)\right)$, with infinitesimal representation

$$d\mu : \mathcal{A} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{sp}(2d, \mathbb{R}) \mapsto -\frac{1}{2i}\begin{pmatrix} \hat{Q} & \hat{P} \end{pmatrix}\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix}\begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix},$$

where $\hat{Q} = x$, $\hat{P} = -i\nabla_x$, $x \in \mathbb{R}^d$ and $id$ is the identity matrix on $\mathbb{R}^d$.

**Theorem 3.28.** *[Fol89, Theorem 4.51] Suppose*

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} = \exp\left(\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} t\right),$$

*where* $\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{sp}(2d, \mathbb{R})$. *For any time $T > 0$ such that when $t \in [0, T]$,*

$\det(D(t)) > 0$, *then*

$$\mu \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} f(x) \tag{3.61}$$
$$= \frac{1}{\det\left(D(t)\right)^{1/2}(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-iS(x,\xi)\right)\hat{f}(-\xi)\,d\xi,\ x \in \mathbb{R}^n,\ t \in [0,T]$$

*where*

$$S(x,\xi) = \frac{-\xi D(t)^{-1}C(t)\xi}{2} + \xi D(t)^{-1}x + \frac{xB(t)D(t)^{-1}x}{2}, \quad x,\xi \in \mathbb{R}^d.$$

### 3.6.4 Global Well-posedness

We establish a global well-posedness result for Equation (2.9) when

$$\left\|\mathfrak{h}^{1/2}\Gamma_0\mathfrak{h}^{1/2}\right\|_{tr} < \infty, \quad \Gamma_0^* = \Gamma_0, \quad \Gamma_0 \geq 0 \text{ and } w(x) = \frac{1}{|x|}.$$

The associated total energy is

$$\mathcal{E}_{HF}\left(\Gamma(t)\right) = Tr\left(\mathfrak{h}^{1/2}\Gamma(t)\mathfrak{h}^{1/2}\right) + \frac{1}{2}\int_{\mathbb{R}^3}\left(\rho_\Gamma * V\right)(t,x)\rho_\Gamma(t,x)\,dx. \tag{3.62}$$

The outline of the proof is that we first establish two local well-posedness results for Equation (2.9): one is at the energy level and another one is for smooth data. Then we verify the conservation law of the total energy for smooth data and use a limiting argument to pass the law to the energy level. Finally, the global well-posedness follows from the conservation of energy. All estimates involved are based on time-independent arguments.

Note that $\mathfrak{h} = L^* L$, where

$$L = \left(-i\partial_{x^1} + \frac{b}{2}x^2, -i\partial_{x^2} - \frac{b}{2}x^1, -i\partial_{x^3}\right)$$

and $x = (x^1, x^2, x^3)$, and the covariant derivative $L$ is metric. The pointwise Kato's inequality holds

$$|\nabla |f|| \lesssim |Lf|\,. \tag{3.63}$$

In addition

$$\left\|\mathfrak{h}^{1/2} f\right\|^2_{L^2(\mathbb{R}^3)} = \|Lf\|^2_{L^2(\mathbb{R}^3)} = \|Df\|^2_{L^2(\mathbb{R}^3)} + \|\partial_{x^3} f\|^2_{L^2(\mathbb{R}^3)} + b\,\|f\|^2_{L^2(\mathbb{R}^3)}\,.$$

Let us define the following operator norms for the discussion

$$\|\Gamma\|_{\mathcal{L}^{s,p}_{\mathfrak{h}}} := \left\|\mathfrak{h}^{s/2}\Gamma\mathfrak{h}^{s/2}\right\|_{\mathcal{L}^p} = \left(Tr\left|\mathfrak{h}^{s/2}\Gamma\mathfrak{h}^{s/2}\right|^p\right)^{1/p} \tag{3.64}$$

where $s \geq 0$, $1 \leq p \leq \infty$ and $\mathcal{L}^p$ is the $p$-th Schatten norm.

1. The local well-posedness at the energy level.

To deal with the nonlinear term in Equation (2.9), we first show a bilinear estimate for functions, then generalize it to operators.

**Proposition 3.29.**

$$\left\| \mathfrak{h}^{1/2} \left( \left( |\phi_1|^2 * V \right) \phi_2 \right) \right\|_{L^2} \lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2 \left\| \mathfrak{h}^{1/2} \phi_2 \right\|_{L^2}. \tag{3.65}$$

*Proof.* Applying the Hölder inequality,

$$\left\| \mathfrak{h}^{1/2} \left( \left( |\phi_1|^2 * V \right) \phi_2 \right) \right\|_{L^2} \lesssim \left\| |\phi_1|^2 * V \right\|_{L^\infty} \left\| h^{1/2} \phi_2 \right\|_{L^2} + \left\| |\nabla_x| \left( |\phi_1|^2 * V \right) \right\|_{L^3} \|\phi_2\|_{L^6},$$

while

$$\begin{aligned} \left( |\phi_1|^2 * V \right)(x) &= \int_{\mathbb{R}^3} |\phi_1|^2 (x-y) V(y) \, dy \\ &\lesssim \int_{\mathbb{R}^3} \left| |\nabla_y|^{1/2} |\phi_1| (x-y) \right|^2 dy \quad \text{(by the Hardy's inequality)} \\ &\leq \int_{\mathbb{R}^3} \left( \left| |\nabla| |\phi_1| \right|^2 (x) + |\phi_1|^2 (x) \right) dx \\ &\lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2, \quad \text{(by the inequality (3.63))} \end{aligned}$$

and by the inequality (3.63), the Sobolev inequality and the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} &\|\phi_2\|_{L^6} \lesssim \| |\phi_2| \|_{H^1} \lesssim \left\| \mathfrak{h}^{1/2} \phi_2 \right\|_{L^2}, \\ &\left\| |\nabla_x| \left( |\phi_1|^2 * V \right) \right\|_{L^3} = \left\| |\phi_1|^2 * \left( |\nabla| V \right) \right\|_{L^3} \lesssim \left\| |\phi_1|^2 \right\|_{L^{3/2}} = \|\phi_1\|_{L^3}^2 \lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2, \end{aligned}$$

we obtain the desired estimate,

$$\left\| \mathfrak{h}^{1/2} \left( \left( |\phi_1|^2 * V \right) \phi_2 \right) \right\|_{L^2} \lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2 \left\| \mathfrak{h}^{1/2} \phi_2 \right\|_{L^2}.$$

$\square$

**Proposition 3.30.** *Suppose $\Gamma_1$ and $\Gamma_2$ are self-adjoint,*

$$\|[\rho_{\Gamma_1} * V, \Gamma_2]\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}} \lesssim \|\Gamma_1\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}} \|\Gamma_2\|_{\mathcal{L}^{1,1}_{h\mathfrak{h}}} \tag{3.66}$$

*Proof.* Since $\Gamma_j$ is self-adjoint and $\|\Gamma_j\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}} < \infty$ for $j = 1, 2$, there are orthonormal bases $\{f_{k,j}\}_{k=1}^{\infty}$ $j = 1, 2$, such that

$$\left(\mathfrak{h}^{1/2}\Gamma_j\mathfrak{h}^{1/2}\right)(x, y) = \sum_{k=1}^{\infty} \lambda_{k,j} f_{k,j}(x)\bar{f}_{k,j}(y).$$

Then

$$\Gamma_j(x, y) = \sum_{k=1}^{\infty} \lambda_{k,j} \left(\mathfrak{h}^{-1/2} f_{k,j}\right)(x) \left(\overline{\mathfrak{h}^{-1/2} f}_{k,j}\right)(y),$$

and by the Minkowski's inequality,

$$\begin{aligned}
\|(\rho_{\Gamma_1} * V)\,\Gamma_2\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}} &= \left\|\mathfrak{h}_x^{1/2}\left((\rho_{\Gamma_1} * V)(x) \sum_{k=1}^{\infty} \lambda_{k,2} \left(\mathfrak{h}^{-1/2} f_{k,2}\right)(x) \left(\bar{f}_{k,2}\right)(y)\right)\right\|_{tr} \\
&\le \sum_{k=1}^{\infty} |\lambda_{k,2}| \left\|h_x^{1/2}\left((\rho_{\Gamma_1} * V)(x) \left(\mathfrak{h}^{-1/2} f_{k,2}\right)(x) \left(\bar{f}_{k,2}\right)(y)\right)\right\|_{tr} \\
&\le \sum_{k=1}^{\infty} |\lambda_{k,2}| \left\|h_x^{1/2}\left((\rho_{\Gamma_1} * V)(x) \left(\mathfrak{h}^{-1/2} f_{k,2}\right)(x)\right)\right\|_{L^2_x} \\
&\le \sum_{k=1}^{\infty} |\lambda_{k,2}| \sum_{l=1}^{\infty} |\lambda_{l,1}| \left\|h^{1/2}\left(\left(\left|\mathfrak{h}^{-1/2} f_{l,1}\right|^2 * V\right) \mathfrak{h}^{-1/2} f_{k,2}\right)\right\|_{L^2} \\
&\lesssim \sum_{k=1}^{\infty} |\lambda_{k,2}| \sum_{l=1}^{\infty} |\lambda_{l,1}| \, \|f_{l,1}\|_{L^2}^2 \, \|f_{k,2}\|_{L^2} \qquad \text{(by Proposition 3.29)} \\
&\le \sum_{k=1}^{\infty} |\lambda_{k,2}| \sum_{l=1}^{\infty} |\lambda_{l,1}|.
\end{aligned}$$

The other term $\|\Gamma_2\,(\rho_{\Gamma_1} * V)\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}}$ can be estimated in the same way. $\square$

Based on Proposition 3.30, we obtain the following local well-posedness result as an application of the contraction mapping principle.

**Theorem 3.31.** *For any initial data $\|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}} < \infty$ and $\Gamma_0^* = \Gamma_0$, Equation (2.9) has a mild solution in the Banach space $\mathbf{N}_{1T}$, where the norm $\mathbf{N}_{1T}$ is defined as*

$$\|\Gamma(t)\|_{\mathbf{N}_{1T}} := \|\Gamma(t)\|_{L^\infty\left([0,T];\mathcal{L}_{\mathfrak{h}}^{1,1}\right)}, \tag{3.67}$$

*while the existence time $T$ depends on $\|\mathfrak{h}^{1/2}\Gamma_0\mathfrak{h}^{1/2}\|_{tr}$. To be more precise, the solution $\Gamma(t) \in C^0\left([0,T];\mathcal{L}_{\mathfrak{h}}^{1,1}\right)$.*

2. The local well-posedness for smooth data.

Similarly as Step 1, we first show a bilinear estimate for functions, then generalize it to operators.

**Proposition 3.32.**

$$\left\|h\left(\left(|\phi_1|^2 * V\right)\phi_2\right)\right\|_{L^2} \lesssim \left\|\mathfrak{h}^{1/2}\phi_1\right\|_{L^2}^2 \|h\phi_2\|_{L^2} \tag{3.68}$$

*Proof.* A direct computation shows

$$\begin{aligned}
& h\left(\left(|\phi_1|^2 * V\right)\phi_2\right) \\
=& -\Delta\left(|\phi_1|^2 * V\right)\phi_2 + \left(|\phi_1|^2 * V\right)h\phi_2 \\
& + \underbrace{\left(-2\partial_{\bar z}\left(|\phi_1|^2 * V\right)\right)D^*\phi_2 + \left(2\partial_z\left(|\phi_1|^2 * V\right)\right)D\phi_2 - 2\partial_{x^3}\left(|\phi_1|^2 * V\right)\partial_{x^3}\phi_2}_{\text{first-order terms}}.
\end{aligned}$$

By the proof of Proposition 3.29,

$$\left\| \left( |\phi_1|^2 * V \right) h\phi_2 \right\|_{L^2} \leq \left\| |\phi_1|^2 * V \right\|_{L^\infty} \| h\phi_2 \|_{L^2} \lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2 \| h\phi_2 \|_{L^2} ,$$

and

$$\begin{aligned} \| \text{first-order terms} \|_{L^2} &\lesssim \left\| |\nabla| \left( |\phi_1|^2 * V \right) \right\|_{L^3} \left( \| D^* \phi_2 \|_{L^6} + \| D\phi_2 \|_{L^6} + \| \partial_{x^3} \phi_2 \|_{L^6} \right) \\ &\lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2 \| h\phi_2 \|_{L^2} . \end{aligned}$$

Analyzing $-\Delta \left( |\phi_1|^2 * V \right) \phi_2$, by the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality,

$$\begin{aligned} \left\| -\Delta \left( |\phi_1|^2 * V \right) \phi_2 \right\|_{L^2} &\leq \left\| \left( |\nabla| |\phi_1|^2 \right) * \left( |\nabla| w \right) \right\|_{L^3} \| \phi_2 \|_{L^6} \\ &\lesssim \left\| \nabla |\phi_1|^2 \right\|_{L^{3/2}} \| \phi_2 \|_{L^6} \\ &\lesssim \| \nabla |\phi_1| \|_{L^2} \| \phi_1 \|_{L^6} \| \phi_2 \|_{L^6} \\ &\lesssim \left\| \mathfrak{h}^{1/2} \phi_1 \right\|_{L^2}^2 \left\| \mathfrak{h}^{1/2} \phi_2 \right\|_{L^2} . \end{aligned}$$

$\square$

Using the same argument in Proposition 3.30, we generalize Proposition 3.32 to operators.

**Proposition 3.33.** *Suppose $\Gamma_1$ and $\Gamma_2$ are self-adjoint,*

$$\left\| \left[ \rho_{\Gamma_1} * V, \Gamma_2 \right] \right\|_{\mathcal{L}^{2,1}_{\mathfrak{h}}} \lesssim \| \Gamma_1 \|_{\mathcal{L}^{1,1}_{\mathfrak{h}}} \| \Gamma_2 \|_{\mathcal{L}^{2,1}_{\mathfrak{h}}} . \tag{3.69}$$

**Theorem 3.34.** *For any initial data $\|\Gamma_0\|_{\mathcal{L}^{2,1}} < \infty$ and $\Gamma_0^* = \Gamma_0$, Equation (2.9) has a mild solution in the Banach space $\mathbf{N}_{2T}$, where the norm $\mathbf{N}_{2T}$ is defined as*

$$\|\Gamma(t)\|_{\mathbf{N}_{2T}} := \|\Gamma(t)\|_{L^\infty([0,T];\mathcal{L}^{2,1})}, \tag{3.70}$$

*while $I_T = [0,T]$ and the existence time $T$ depends on $\|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}$. More precisely, the solution $\Gamma(t) \in C^0\left([0,T], \mathcal{L}_{\mathfrak{h}}^{2,1}\right) \cap C^1\left([0,T], \mathcal{L}^1\right)$*

*Proof.* Based on Proposition 3.33, we use the contraction mapping principle to obtain the local well-posedness result.

To show the existence time $T$ depends on $\|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}$, consider the integral form of the solution $\Gamma(t)$

$$\Gamma(t) = e^{-i\mathfrak{h}t}\Gamma_0 e^{i\mathfrak{h}t} - i\int_0^t e^{-i\mathfrak{h}(t-\tau)}\left[\rho_{\Gamma(\tau)} * V, \Gamma(\tau)\right] e^{i\mathfrak{h}(t-\tau)}\, d\tau,$$

then by the Minkowski's inequality,

$$\begin{aligned}\|\Gamma(t)\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} &\le \left\|e^{-i\mathfrak{h}t}\Gamma_0 e^{i\mathfrak{h}t}\right\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} + \int_0^t \left\|\left[\rho_{\Gamma(\tau)} * V, \Gamma(\tau)\right]\right\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} d\tau \\ &\le \|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} + C\left(\sup_{\tau\in I_T}\|\Gamma(\tau)\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}\right)\int_0^t \|\Gamma(\tau)\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}}\, d\tau \qquad \text{(Proposition 3.33)},\end{aligned}$$

where $C$ is a constant. Using the Grönwall's inequality, for $0 \le t \le T$,

$$\|\Gamma(t)\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} \le \|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{2,1}} \exp\left(Ct\sup_{\tau\in I_T}\|\Gamma(\tau)\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}\right).$$

Since Theorem 3.31 says that the existence $T$ depends on $\|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}$, with the above estimate, so is the case for Theorem 3.34. By the semi-group theory, the solution $\Gamma(t) \in C^0\left([0,T], \mathcal{L}_{\mathfrak{h}}^{2,1}\right) \cap C^1\left([0,T], \mathcal{L}^1\right)$. ☐

3. The conservation law.

We first verify the conservation law of energy for smooth data, then pass it to the energy level by the limiting argument.

**Proposition 3.35.** *Suppose that $\Gamma(t) \in C^0\left([0,T], \mathcal{L}_{\mathfrak{h}}^{2,1}\right) \cap C^1\left([0,T], \mathcal{L}^1\right)$ is a solution to Equation (2.9), then the total energy (3.62) $\mathcal{E}_{HF}\left(\Gamma(t)\right)$ is conserved for $t \in [0,T]$.*

*Proof.* The trick is to express (3.62) in the following way

$$\mathcal{E}_{HF}\left(\Gamma\right) = Tr\left(\mathfrak{h}\Gamma\right) + \frac{1}{2} Tr\left(\left(\rho_{\Gamma} * V\right)\Gamma\right) = Tr\left(\Gamma\mathfrak{h}\right) + \frac{1}{2} Tr\left(\Gamma\left(\rho_{\Gamma} * V\right)\right),$$

and use the mild formulation

$$\Gamma(t) = e^{-i\mathfrak{h}t}\Gamma_0 e^{i\mathfrak{h}t} - i\int_0^t e^{-i\mathfrak{h}(t-\tau)}\left[\rho_{\Gamma(\tau)} * V, \Gamma(\tau)\right] e^{i\mathfrak{h}(t-\tau)}\, d\tau.$$

Taking the time derivative

$$
\begin{aligned}
&\frac{d\,\mathcal{E}_{HF}(\Gamma(t))}{dt}\\
=&-i\,Tr\left(\mathfrak{h}e^{-i\mathfrak{h}t}\Gamma_0 e^{i\mathfrak{h}t}h\right)-\int_0^t d\tau\,Tr\left(\mathfrak{h}e^{-i\mathfrak{h}(t-\tau)}\left[\rho_{\Gamma(\tau)}*V,\Gamma(\tau)\right]e^{i\mathfrak{h}(t-\tau)}\mathfrak{h}\right)\\
&+i\,Tr\left(\mathfrak{h}e^{-i\mathfrak{h}t}\Gamma_0 e^{i\mathfrak{h}t}\mathfrak{h}\right)+\int_0^t d\tau\,Tr\left(\mathfrak{h}e^{-i\mathfrak{h}(t-\tau)}\left[\rho_{\Gamma(\tau)}*V,\Gamma(\tau)\right]e^{i\mathfrak{h}(t-\tau)}\mathfrak{h}\right)\\
&-i\,Tr\left(\left[\rho_{\Gamma(t)}*V,\Gamma(t)\right]\mathfrak{h}\right)+Tr\left(\dot{\Gamma}(t)\left(\rho_{\Gamma(t)}*V\right)\right)\\
=&-i\,Tr\left(\left[\rho_{\Gamma(t)}*V,\Gamma(t)\right]\mathfrak{h}\right)-i\,Tr\left(\left[\mathfrak{h}+\rho_{\Gamma(t)}*V,\Gamma(t)\right]\left(\rho_{\Gamma(t)}*V\right)\right)\\
=&0\qquad\text{(cyclicity of Tr)}.
\end{aligned}
$$

By the fundamental theorem of calculus, $\mathcal{E}_{HF}\left(\Gamma(t)\right)=\mathcal{E}_{HF}\left(\Gamma_0\right)$ for $0\le t\le T$. □

For any initial data $\Gamma_0$ at the energy level, i.e.

$$
\|\Gamma_0\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}}<\infty,\quad \Gamma_0^*=\Gamma_0,
$$

there exists a sequence $\{\Gamma_{0,k}\}_{k=1}^{\infty}\subset\mathcal{L}^{2,1}_{\mathfrak{h}}$ such that

$$
\lim_{k\to\infty}\|\Gamma_{0,k}-\Gamma_0\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}}=0.
$$

Denote the solution of Equation (2.9) associated to the initial data $\Gamma_{0,k}$ by $\Gamma_k(t)$. Since the existence time of $\Gamma_k(t)$ depends on $\|\Gamma_{0,k}\|_{\mathcal{L}^{1,1}_{\mathfrak{h}}}$ (Theorem 3.34), there is a uniform time $T$ such that all solutions $\Gamma_k(t)$ exist in the sense of Theorem 3.34. By the continuous dependence on initial data (from Theorem 3.31), for any $0 \le t \le T$,

$$\lim_{k\to\infty} \|\Gamma_k(t) - \Gamma(t)\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}} = 0.$$

While the total energy $\mathcal{E}_{HF}$ is continuous with respect to the norm $\mathcal{L}_{\mathfrak{h}}^{1,1}$, by Proposition 3.35,

$$\mathcal{E}_{HF}(\Gamma(t)) = \lim_{k\to\infty} \mathcal{E}_{HF}(\Gamma_k(t)) = \lim_{k\to\infty} \mathcal{E}_{HF}(\Gamma_{0,k}) = \mathcal{E}_{HF}(\Gamma_0). \tag{3.71}$$

4. The global well-posedness at the energy level.

Note that when the initial data $\Gamma_0$ is non-negative, i.e. it satisfies the operator inequality $\Gamma_0 \ge 0$, the condition of being non-negative is preserved under Equation (2.9). Thus $Tr(\mathfrak{h}^{1/2}\Gamma(t)\mathfrak{h}^{1/2}) = \|\Gamma(t)\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}$ and the energy $\mathcal{E}_{HF}(\Gamma(t)) \sim \|\Gamma(t)\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}}$. Using the conservation law (3.71), we improve the local well-posedness result Theorem 3.31 to the following global statement.

**Theorem 3.36.** *Suppose that the initial data $\Gamma_0$ satisfies*

$$\|\Gamma_0\|_{\mathcal{L}_{\mathfrak{h}}^{1,1}} < \infty, \quad \Gamma_0^* = \Gamma_0, \quad \Gamma_0 \ge 0,$$

*then Equation (2.9) has a global mild solution $\Gamma(t) \in C^0\left([0,\infty), \mathcal{L}_{\mathfrak{h}}^{1,1}\right)$.*

# Chapter 4: Global Well-Posedness for Bogoliubov-de Gennes Equations

The chapter is organized as: In Section 3, we prove the local well-posedness result Theorem 2.15. In Section 4, we consider the Bogoliubov-de Gennes equations with smooth and compactly supported potential. The regularity of initial data can be preserved by smooth potential case. We prove the smooth potential version Theorem 4.18 of Theorem 2.16. In Section 5, we assemble results for the smooth potential case and the local case, and use a limiting argument to prove the global result Theorem 2.16. In the appendix, we prove two propositions which are used in Section 3 and Section 4 respectively: the Morrey's inequality for Banach spaces and the property of the Bogoliubov-de Gennes equations with smooth potential that the spectrum of the generalized one-particle density matrix does not change along the time evolution.

## 4.1 Preliminary

Note that the nonlinear terms in Equation (2.39) and (2.40) are quadratic maps of $\Gamma(t)$ and $\Lambda(t)$. For simplicity of notations, we define two bilinear maps based on $F_1(t;v)$ and $F_2(t;v)$ and use a state $\omega(t)$ to refer to a pair of functions

$(\Gamma(t), \Lambda(t))$,

$$B_1(\omega_1, \omega_2; v) := [v * \rho_{\Gamma_1}, \Gamma_2] - [\Gamma_1, \Gamma_2]_v + [\Lambda_1, \Lambda_2^*]_v \tag{4.1}$$

$$B_2(\omega_1, \omega_2; v) := [v * \rho_{\Gamma_1}, \Lambda_2] - [\Gamma_1, \Lambda_2]_{v,+} - [\Lambda_1, \bar{\Gamma}_2]_{v,+}. \tag{4.2}$$

where $(\Gamma_j, \Lambda_j)$ is associated with state $\omega_j$, $j = 1, 2$. $(\Gamma_j, \Lambda_j)$ may not satisfy Condition (2.37) and the two bilinear maps are defined for pairs of general functions.

## 4.2 Derivation of Equations

In this section, we consider a pure quasi-free state and derive the Bogoliubov-de Gennes equations as an effective dynamics of the Many-body problem. The derivation is in the same spirit of [GM13, GM17]. Let $e^{-B_0}|0\rangle$ be in the Fock space $\mathcal{F}_a$, where $e^{-B_0}$ denotes the unitary implementation of a Bogoliubov transform. This state is quasi-free and all pure quasi-free states with finite expected number of particles are in this form [Sol14]. Consider the Schrödinger equation of state $e^{-B_0}|0\rangle$ in the Fock space $\mathcal{F}_a$,

$$i\,\partial_t \Psi_t = \hat{H}\,\Psi_t, \quad \Psi_0 = \mathrm{e}^{-B_0}\,|0\rangle\,, \tag{4.3}$$

The solution to the Schrödinger equation is $e^{-it\hat{H}}e^{-B_0}|0\rangle$. Our goal is to derive an equation to describe the solution $e^{-it\hat{H}}e^{-B_0}|0\rangle$ effectively local in time. An approach is to find $e^{-B_t}|0\rangle$ such that

$$\left\| e^{-it\hat{H}}e^{-B_0}|0\rangle - e^{-B_t}|0\rangle \right\|_{\mathcal{F}_a} = \left\| e^{B_t}e^{-it\hat{H}}e^{-B_0}|0\rangle - |0\rangle \right\|_{\mathcal{F}_a}$$

is minimal, which is equivalent to study $\psi_t = e^{B_t} e^{-it\hat{H}} e^{-B_0} |0\rangle$. $\psi_t$ satisfies the evolution equation

$$i\,\partial_t \psi_t = i\left(\partial_t e^{B_t} e^{-B_t}\right)\psi_t + e^{B_t}\hat{H}e^{-B_t}\psi_t, \quad \psi_0 = |0\rangle\,. \tag{4.4}$$

Denote the reduced Hamiltonian $\hat{H}_{red} = i\left(\partial_t e^{B_t} e^{-B_t}\right) + e^{B_t}\hat{H}e^{-B_t}$. For short time, $\psi_t$ is controlled by

$$\hat{H}_{red}\,|0\rangle = (X_0, X_1, X_2, X_3, X_4, 0, 0, \ldots), \tag{4.5}$$

where $X_1 = 0$. The correlation functions of $e^{-B_t}|0\rangle$ are

$$L_{m,n} \coloneqq \left\langle a_{y_1} \cdots a_{y_m} e^{-B_t}|0\rangle\,, a_{x_1} \cdots a_{x_n} e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} = \left\langle |0\rangle\,, e^{B_t} P_{m,n} e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a}, \tag{4.6}$$

where $P_{m,n} = a^\dagger_{y_1} \cdots a^\dagger_{y_m} \cdot a_{x_1} \cdots a_{x_n}$. A Fock state is determined by its correlation functions and we will derive time evolution equations for correlation functions.

$e^{-B_t}$ is an unitary implementable Bogoliubov transform and corresponds to a matrix

$$\begin{pmatrix} P_t(x,y) & Q_t(x,y) \\ \bar{Q}_t(x,y) & \bar{P}_t(x,y) \end{pmatrix} \tag{4.7}$$

and an adjoint action on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$

$$e^{B_t} a^\dagger_x e^{-B_t} = \int dy \left(P_t(y,[x]) a^\dagger_y + \bar{Q}_t(y,[x]) a_y\right) \tag{4.8}$$

$$e^{B_t} a_x e^{-B_t} = \int dy \left(\bar{P}_t(y,[x]) a_y + Q_t(y,[x]) a^\dagger_y\right). \tag{4.9}$$

Then correlation functions are

$$
\begin{aligned}
& L_{m,n}(y_1, \dots, y_m; x_1, x_2 \dots, x_n) \\
= & \left\langle |0\rangle, e^{B_t} P_{m,n} e^{-B_t} |0\rangle \right\rangle_{\mathcal{F}_a} \\
= & \left\langle |0\rangle, \prod_{j=1}^{m} \int dz_j \left( P_t(z_j, [y_j]) a_{z_j}^\dagger + \bar{Q}_t(z_j, [y_j]) a_{z_j} \right) \right. \\
& \left. \prod_{l=1}^{n} \int dz_l \left( \bar{P}_t(z_l, [z_l]) a_{z_l} + Q_t(z_l, [x_l]) a_{x_l}^\dagger \right) |0\rangle \right\rangle_{\mathcal{F}_a} .
\end{aligned}
$$

**Lemma 4.1.** *Impose the assumption $X_2 = 0$ and $\hat{H}^* = \hat{H}$, then*

$$
\left\langle |0\rangle, \left[ \hat{H}_{red}, e^{B_t} P_{m,n} e^{-B_t} \right] |0\rangle \right\rangle_{\mathcal{F}_a} = 0
$$

*where $(m, n) = (2, 0), (0, 2), (1, 1)$.*

*Proof.* Compute directly

$$
\begin{aligned}
& \left\langle |0\rangle, \left[ \hat{H}_{red}, e^{B_t} P_{m,n} e^{-B_t} \right] |0\rangle \right\rangle_{\mathcal{F}_a} \\
= & \left\langle \hat{H}_{red} |0\rangle, e^{B_t} P_{m,n} e^{-B_t} |0\rangle \right\rangle_{\mathcal{F}_a} - \left\langle |0\rangle, e^{B_t} P_{m,n} e^{-B_t} \hat{H}_{red} |0\rangle \right\rangle_{\mathcal{F}_a} \\
= & \left\langle X_3 + X_4, e^{B_t} P_{m,n} e^{-B_t} |0\rangle \right\rangle_{\mathcal{F}_a} - \left\langle |0\rangle, e^{B_t} P_{m,n} e^{-B_t} (X_3 + X_4) \right\rangle_{\mathcal{F}_a} .
\end{aligned}
$$

When $(m, n) = (2, 0), (0, 2), (1, 1)$, $e^{B_t} P_{m,n} e^{-B_t}$ can at most create or annihilate two particles. The above quantity must vanish. $\square$

In the end, we derive the Bogoliubov-de Gennes equations by imposing the condition that $X_2$ of $\hat{H}_{red} |0\rangle$ is zero.

**Proposition 4.2.** *Let* $X_2 = 0$ *in (4.5),* $\Gamma(t,x,y) = L_{1,1}(t,y,x)$ *and* $\Lambda(t,x,y) = L_{0,2}(t,y,x)$*, the Bogoliubov-de Gennes equations are*

$$
\begin{aligned}
&(i\,\partial_t + \Delta_x + \Delta_y - v(x-y))\,\Lambda(t,x,y)\\
=&\int dz\,(v(x-z)+v(y-z))\,(\Gamma(t,z,z)\Lambda(x,y) - \Lambda(t,x,z)\Gamma(t,y,z)\\
&+\Gamma(t,x,z)\Lambda(t,y,z))\\
&(i\,\partial_t + \Delta_x - \Delta_y)\,\Gamma(t,x,y)\\
=&\int dx\,(v(x-z)-v(y-z))\,(\Gamma(t,x,x)\Gamma(t,x,y) + \Lambda^*(t,z,y)\Lambda(t,x,z)\\
&-\Gamma(t,z,y)\Gamma(t,x,z))\,.
\end{aligned}
$$

*Proof.* Recall that

$$L_{0,2}(t,x_1,x_2) = \left\langle |0\rangle\,, \left(\int dy_1 dy_2\,\bar{P}_t(y_1,[x_1])a_{y_1}Q_t(y_2,[x_2])a^\dagger_{y_2}\right)|0\rangle\right\rangle_{\mathcal{F}_a}\,,$$

and

$$L_{1,1}(t,x_1;x_2) = \left\langle |0\rangle\,, \left(\int dy_1 dy_2\,\bar{Q}_t(y_1,[x_1])a_{y_1}Q_t(y_2,[x_2])a^\dagger_{y_2}\right)|0\rangle\right\rangle_{\mathcal{F}_a}\,.$$

Compute commutators

$$
\begin{aligned}
&[\mathcal{V}, a^\dagger_{x_1}]\\
=&\int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y a_x a^\dagger_{x_1} - a^\dagger_{x_1} \int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y a_x\\
=&\int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y \left(-a^\dagger_{x_1} a_x + \delta(x-x_1)\right) - a^\dagger_{x_1} \int dxdy\, v_N(x-y) a^\dagger_x a^\dagger_y a_y a_x\\
=&\int dy\, v(y-x_1) a^\dagger_{x_1} a^\dagger_y a_y - \int dxdy\, v(x-y) a^\dagger_x a^\dagger_y \left(-a^\dagger_{x_1} a_y + \delta(x_1-y)\right) a_x\\
&- a^\dagger_{x_1} \int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y a_x\\
=&\int dy\, v(y-x_1) a^\dagger_{x_1} a^\dagger_y a_x - \int dx\, v(x-x_1) a^\dagger_x a^\dagger_{x_1} a_x + a^\dagger_{x_1} \int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y a_x\\
&- a^\dagger_{x_1} \int dxdy\, v(x-y) a^\dagger_x a^\dagger_y a_y a_x\\
=&2a^\dagger_{x_1} \int dx\, v(x-x_1) a^\dagger_x a_x,
\end{aligned}
$$

apply the adjoint operator

$$[\mathcal{V}, a_{x_1}] = -\int dx v(x-x_1) a^\dagger_x a_x a_{x_1}.$$

Using the identities

$$[XY, Z] = X[Y,Z] + [X,Z]Y, \quad [X, YZ] = [X,Y]Z + Y[X,Z],$$

we have

$$
\begin{aligned}
&[\mathcal{V}, a^\dagger_{x_1} a_{x_2}] \\
=&[\mathcal{V}, a^\dagger_{x_1}] a_{x_2} + a^\dagger_{x_1} [\mathcal{V}, a_{x_2}] \\
=&a^\dagger_{x_1} \int dx\, v(x - x_1) a^\dagger_x a_x a_{x_2} - a^\dagger_{x_1} \int dx\, v(x - x_2) a^\dagger_x a_x a_{x_2},
\end{aligned}
$$

and

$$
\begin{aligned}
&[\mathcal{V}, a_{x_1} a_{x_2}] \\
=&[\mathcal{V}, a_{x_1}] a_{x_2} + a_{x_1} [\mathcal{V}, a_{x_2}] \\
=&-\int dx v(x - x_1) a^\dagger_x a_x a_{x_1} a_{x_2} - a_{x_1} \int dx v(x - x_2) a^\dagger_x a_x a_{x_2} \\
=&-v(x_1 - x_2) a_{x_1} a_{x_2} - \int dx v(x - x_1) a^\dagger_x a_x a_{x_1} a_{x_2} - \int dx v(x - x_2) a^\dagger_x a_x a_{x_1} a_{x_2}.
\end{aligned}
$$

Compute time derivatives

$$
\begin{aligned}
&\partial_t \left( e^{B_t} P_{m,n} e^{-B_t} \right) \\
=&\left( \partial_t e^{B_t} e^{-B_t} \right) e^{B_t} P_{m,n} e^{-B_t} + e^{B_t} P_{m,n} e^{-B_t} \left( e^{B_t} \partial_t e^{-B_t} \right) \\
=&\left( \partial_t e^{B_t} e^{-B_t} \right) e^{B_t} P_{m,n} e^{-B_t} - e^{B_t} P_{m,n} e^{-B_t} \left( \partial_t e^{B_t} e^{-B_t} \right) \\
=&\left[ \partial_t e^{B_t} e^{-B_t}, e^{B_t} P_{m,n} e^{-B_t} \right] \\
=&\left[ -i\, \hat{H}_{red} + i\, e^{B_t} \hat{H} e^{-B_t}, e^{B_t} P_{m,n} e^{-B_t} \right] \\
=&-i \left[ \hat{H}_{red}, e^{B_t} P_{m,n} e^{-B_t} \right] + i\, e^{B_t} \left[ \hat{H}, P_{m,n} \right] e^{-B_t}.
\end{aligned}
$$

Using Lemma 4.1,

$$
\begin{aligned}
& i\frac{\partial L_{0,2}}{\partial t}(t,x_1,x_2) \\
&= -\left\langle |0\rangle, e^{B_t}\left[-\Delta, P_{0,2}\right] e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} - \left\langle |0\rangle, e^{B_t}\left[\mathcal{V}, P_{0,2}\right] e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} \\
&= \left\langle |0\rangle, e^{B_t}\left(a_{x_1}a_{x_2}(-\Delta)^*_{x_2} + a_{x_1}(-\Delta)^*_{x_1}a_{x_2}\right) e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} - \left\langle |0\rangle, e^{B_t}\left[\mathcal{V}, P_{0,2}\right] e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a},
\end{aligned}
$$

where

$$
\begin{aligned}
& \left\langle |0\rangle, e^{B_t}\left(a_{x_1}a_{x_2}(-\Delta)^*_{x_2} + a_{x_1}(-\Delta)^*_{x_1}a_{x_2}\right) e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} \\
&= \left\langle |0\rangle, \int dy_1 dy_2 \left(\bar{P}_t(y_1,[x_1])a_{y_1}Q_t(y_2,[x_2])a^\dagger_{y_2}(-\Delta)^*_{x_2}\right.\right. \\
&\quad \left.\left. +\bar{P}_t(y_1,[x_1])a_{y_1}(-\Delta)^*_{x_1}Q_t(y_2,[x_2])a^\dagger_{y_2}\right)|0\rangle\right\rangle_{\mathcal{F}_a} \\
&= \left\langle |0\rangle, \int dy_1 dy_2 \left(\bar{P}_t(y_1,[x_1])a_{y_1}(-\Delta)_{x_2}Q_t(y_2,[x_2])a^\dagger_{y_2}\right.\right. \\
&\quad \left.\left. +(-\Delta)_{x_1}\bar{P}_t(y_1,[x_1])a_{y_1}Q_t(y_2,[x_2])a^\dagger_{y_2}\right)|0\rangle\right\rangle_{\mathcal{F}_a},
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\langle |0\rangle, e^{B_t}\left[\mathcal{V}, P_{0,2}\right] e^{-B_t}|0\rangle\right\rangle_{\mathcal{F}_a} \\
&= -v(x_1-x_2)L_{0,2}(t,x_1,x_2) - \int dx\ \left(v(x-x_1)+v(x-x_2)\right)L_{1,3}(t,x;x,x_1,x_2)
\end{aligned}
$$

We obtain the equation

$$\left(i\frac{\partial}{\partial t} + \Delta_{x_1} + \Delta_{x_2} - v(x_1 - x_2)\right) L_{0,2}(t, x_1, x_2)$$
$$= \int dx \ (v(x - x_1) + v(x - x_2)) \, L_{1,3}(t, x; x, x_1, x_2).$$

Similarly,

$$i\frac{\partial L_{1,1}}{\partial t}(t, x_1; x_2)$$
$$= \left\langle |0\rangle \, , e^{B_t} \left(a^\dagger_{x_1} a_{x_2} (-\Delta)^*_{x_2} - a^\dagger_{x_1} H_{x_1} a_{x_2}\right) e^{-B_t} \, |0\rangle \right\rangle_{\mathcal{F}_a} - \left\langle |0\rangle \, , e^{B_t} \left[\mathcal{V}, P_{1,1}\right] e^{-B_t} \, |0\rangle \right\rangle_{\mathcal{F}_a}$$

where

$$\left\langle |0\rangle \, , e^{B_t} \left(a^\dagger_{x_1} a_{x_2} (-\Delta)^*_{x_2} - a^\dagger_{x_1} (-\Delta)_{x_1} a_{x_2}\right) e^{-B_t} \, |0\rangle \right\rangle_{\mathcal{F}_a}$$
$$= \Big\langle |0\rangle \, , \int dy_1 dy_2 \left(\bar{Q}_t(y_1, [x_1]) a_{y_1} Q_t(y_2, [x_2]) a^\dagger_{y_2} (-\Delta)^*_{x_2}\right.$$
$$\left. - \bar{Q}_t(y_1, [x_1]) a_{y_1} (-\Delta)_{x_1} Q_t(y_2, [x_2]) a^\dagger_{y_2}\right) |0\rangle \Big\rangle_{\mathcal{F}_a}$$
$$= \Big\langle |0\rangle \, , \int dy_1 dy_2 \left(\bar{Q}_t(y_1, [x_1]) a_{y_1} (-\Delta)_{x_2} Q_t(y_2, [x_2]) a^\dagger_{y_2}\right.$$
$$\left. - (-\Delta)^*_{x_1} \bar{Q}_t(y_1, [x_1]) a_{y_1} Q_t(y_2, [x_2]) a^\dagger_{y_2}\right) |0\rangle \Big\rangle_{\mathcal{F}_a}$$

and

$$\left\langle |0\rangle \, , e^{B_t} \left[\mathcal{V}, P_{1,1}\right] e^{-B_t} \, |0\rangle \right\rangle_{\mathcal{F}_a}$$
$$= \int dx \ (v(x - x_1) - v(x - x_2)) \, L_{2,2}(t, x_1, x; x, x_2).$$

We obtain the second equation

$$(i\,\partial_t - \Delta_{x_1} + \Delta_{x_2})\, L_{1,1}(t, x_1, x_2) = -\int dx\, (v(x - x_1) - v(x - x_2))\, L_{2,2}(t, x_1, x, x; x_2). \tag{4.10}$$

Compute four-particle correlation functions,

$$\begin{aligned}
&L_{1,3}(t, x_1; x_2, x_3, x_4) \\
=&\Big\langle |0\rangle\,, \int dy_1 dy_2 dy_3 dy_4\, \bar{Q}_t(y_1, [x_1]) a_{y_1} Q_t(y_2, [x_2]) a^\dagger_{y_2} \\
&\quad \bar{P}_t(y_3, [x_3]) a_{y_3} Q_t(y_4, [x_4]) a^\dagger_{y_4} |0\rangle \Big\rangle_{\mathcal{F}_a} \\
&+ \Big\langle |0\rangle\,, \int dy_1 dy_2 dy_3 dy_4\, \bar{Q}_t(y_1, [x_1]) a_{y_1} \bar{P}_t(y_2, [x_2]) a_{y_2} \\
&\quad Q_t(y_3, [x_3]) a^\dagger_{y_3} Q_t(y_4, [x_4]) a^\dagger_{y_4} |0\rangle \Big\rangle_{\mathcal{F}_a} \\
=&L_{1,1}(t, x_1; x_2) L_{0,2}(t, x_3, x_4) - L_{1,1}(t, x_1; x_3) L_{0,2}(t, x_2, x_4) \\
&+ L_{1,1}(t, x_1; x_4) L_{0,2}(t, x_2, x_3)
\end{aligned}$$

and

$$
\begin{aligned}
&L_{2,2}(t,x_1,x_2;x_3,x_4)\\
=&\left\langle |0\rangle \,, \int dy_1dy_2dy_3dy_4\, \bar{Q}_t(y_1,[x_1])a_{y_1}P_t(y_2,[x_2])a^\dagger_{y_2}\right.\\
&\quad \left. \bar{P}_t(y_3,[x_3])a_{y_3}Q_t(y_4,[x_4])a^\dagger_{y_4}\,|0\rangle\right\rangle_{\mathcal{F}_a}\\
&+\left\langle |0\rangle \,, \int dy_1dy_2dy_3dy_4\, \bar{Q}_t(y_1,[x_1])a_{y_1}\bar{Q}_t(y_2,[x_2])a_{y_2}\right.\\
&\quad \left. Q_t(y_3,[x_3])a^\dagger_{y_3}Q_t(y_4,[x_4])a^\dagger_{y_4}\,|0\rangle\right\rangle_{\mathcal{F}_a}\\
=&L^*_{0,2}(t,x_1,x_2)L_{0,2}(t,x_3,x_4)-L_{1,1}(t,x_1;x_3)L_{1,1}(t,x_2;x_4)\\
&+L_{1,1}(t,x_1;x_4)L_{1,1}(t,x_2;x_3).
\end{aligned}
$$

□

## 4.3 Local Well-Posedness Theory

In this section, we prove the local well-posedness Theorem 2.15 by showing that the Duhamel's formulation (2.51) has a fixed point in the solution space (2.49) for sufficiently small $T$. Our strategy is to arrange quantities in (2.49) into three groups

$$
\|\Gamma(t)\|_{L^\infty_t([0,T],\mathcal{L}^1)}+\|\rho_{\Gamma(t)}(x)\|_{L^1_tL^3_x([0,T]\times\mathbb{R}^3)},\ \|\Gamma(t,x,y)\|_{ST^1_T} \text{ and } \|\Lambda(t,x,y)\|_{ST^1_{\epsilon T}},
$$

and consider the linear part $e^{i\Delta t}\Gamma_0e^{-i\Delta t}$ or $e^{i\Delta t}\Lambda_0e^{i\Delta t}$, $v\Lambda$ and the nonlinear part $F_1$ or $F_2$ in (2.51) for each case.

**I** To estimate $\|\Gamma(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)} + \|\rho_{\Gamma(t)}(x)\|_{L_t^1 L_x^3([0,T]\times\mathbb{R}^3)}$: Based on the observation that the linear propagator $e^{i\Delta t}\Gamma_0 e^{-i\Delta t}$ preserves the spectrum of $\Gamma_0$ and $\rho_{\Gamma(t)}$ can be written as a sum of products of two functions if $\Gamma(t)$ is of trace class, using the Strichartz estimate for functions, the linear part $e^{i\Delta t}\Gamma_0 e^{-i\Delta t}$ is controlled by $\|\Gamma_0\|_{\mathcal{L}^1}$. Similarly, the nonlinear part is majorized by $\|F_1(t;v)\|_{L_t^1([0,T],\mathcal{L}^1)}$.

**II** To estimate $\|\Gamma(t,x,y)\|_{ST_T^1}$: We apply the Strichartz estimate for functions valued in a Hilbert space, estimate the linear part by $\|\Gamma_0\|_{H^1}$ and the nonlinear part by $\|F_1(t,x,y;v)\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$.

**III** To estimate $\|\Lambda(t,x,y)\|_{ST_{\epsilon T}^1}$: We could still control the nonlinear part by $\|F_2(t,x,y;v)\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$ and the linear part $e^{i\Delta t}\Lambda_0 e^{i\Delta t}$ by $\|\Lambda_0\|_{H^1}$ as Step II. The singular term $(v\Lambda)(t)$ is treated as a forcing term and we put $\langle\nabla_{x,y}\rangle\left((v\Lambda)(t)\right)$ in the dual Strichartz space $L_t^2 L_{x-y}^{6/5} L_{x+y}^2$. Since $\Lambda(t,x,x)$ vanishes for all $t$ and $x$, using Proposition 4.19 the Morrey's inequality for Banach spaces[1], the singularity $x = 0$ of $|\nabla_x| v(x)$ is mitigated by $\Lambda(t,x,x)$.

Next we elaborate each step in details in the rest of the section. Case I is based on the following lemma.

**Lemma 4.3.** *Let $\Gamma(t)$ be the solution to the linear equation*

$$i\,\partial_t\Gamma(t) = [-\Delta, \Gamma(t)], \qquad \Gamma(t=0) = \Gamma_0 \quad \textit{and} \quad \Gamma_0^* = \Gamma_0.$$

[1]The proof of Proposition 4.19 is essentially the same as the classic case and we prove it in appendix.

*Then* $\Gamma(t)^* = \Gamma(t)$ *and the following estimate holds*

$$\|\Gamma(t)\|_{L_t^\infty(\mathbb{R},\mathcal{L}^1)} + \|\rho_{\Gamma(t)}(x)\|_{L_t^1 L_x^3} \lesssim \|\Gamma_0\|_{\mathcal{L}^1}. \tag{4.11}$$

*Furthermore, if* $\Gamma(t)$ *is the solution to the inhomogenous equation*

$$i\,\partial_t\Gamma(t) = [-\Delta, \Gamma(t)] + F(t), \qquad \Gamma(t=0) = \Gamma_0 \quad \textit{and} \quad \Gamma_0^* = \Gamma_0,$$

*where* $F^*(t) = -F(t)$*, then for any* $T \in \mathbb{R}$*,*

$$\|\Gamma(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)} + \left\|\rho_{\Gamma(t)}\right\|_{L_t^1 L_x^3([0,T]\times\mathbb{R}^3)} \lesssim \|\Gamma_0\|_{\mathcal{L}^1} + \|F(t)\|_{L_t^1([0,T],\mathcal{L}^1)}. \tag{4.12}$$

*Proof.* Let $\Gamma(t)$ be the solution to the linear equation, then $\Gamma(t) = e^{i\Delta t}\Gamma_0 e^{-i\Delta t}$. Since the linear propagator $e^{i\Delta t}\Gamma_0 e^{-i\Delta t}$ preserves the spectrum of $\Gamma_0$, $\|e^{i\Delta t}\Gamma_0 e^{-i\Delta t}\|_{\mathcal{L}^1} = \|\Gamma_0\|_{\mathcal{L}^1}$. To derive the estimate for $\rho_{\Gamma(t)}$, note that $\Gamma_0$ is of trace class and self-adjoint. Then there is an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of $L^2(\mathbb{R}^3)$ such that

$$\Gamma_0(x,y) = \sum_{j=1}^\infty \lambda_j \phi_j(x)\bar{\phi}_j(y),$$

where $\lambda_j$ are singular values of $\Gamma_0$ and $\sum_{j=1}^\infty |\lambda_j| = \|\Gamma_0\|_{\mathcal{L}^1}$. Express the solution $\Gamma(t) = e^{i\Delta t}\Gamma_0 e^{-i\Delta t}$ in terms of the basis

$$e^{i(\Delta_x - \Delta_y)t}\Gamma_0 = \sum_{j=1}^\infty \lambda_j e^{i\Delta_x t}\phi_j(x)\overline{e^{i\Delta_y t}\phi_j}(y),$$

and the collapsing term is $\rho_{\Gamma(t)} = \sum_{j=1}^{\infty} \lambda_j \left|e^{i\Delta_x t}\phi_j\right|^2(x)$. Applying the Endpoint Strichartz estimate and the Hölder inequality,

$$\left\|\rho_{\Gamma(t)}\right\|_{L_t^1 L_x^3} \leq \sum_{j=1}^{\infty} |\lambda_j| \left\|e^{i\Delta_x t}\phi_j\right\|_{L_t^2 L_x^6}^2 \lesssim \sum_{j=1}^{\infty} |\lambda_j| \left\|\phi_j\right\|_{L_x^2}^2 = \|\Gamma_0\|_{\mathcal{L}^1}.$$

Besides, the operator $e^{i\Delta t}$ is unitary and we obtain the estimate (trace theorem)

$$\left\|\rho_{\Gamma(t)}\right\|_{L_t^\infty L_x^1} \leq \sum_{j=1}^{\infty} |\lambda_j| = \|\Gamma_0\|_{\mathcal{L}^1}.$$

When $\Gamma(t)$ is the solution to the inhomogeneous equation, applying the linear estimate (4.11) and the Minkowski inequality to the Duhamel's formulation,

$$\Gamma(t) = e^{i\Delta t}\Gamma_0 e^{-i\Delta t} - \int_0^t ds\, e^{i\Delta(t-s)} iF(s) e^{-i\Delta(t-s)},$$

we obtain (4.12). □

After applying Lemma 4.3 to the $\Gamma$ Equation (2.39) and treating $F_1(t;v)$ as a forcing term, in order to close the fixed point argument, we need to estimate $\|F_1(t;v)\|_{L^1_{[0,T]}\mathcal{L}^1}$ by $\|\Gamma(t)\|_{N_{1T}}$ and $\|\Lambda(t)\|_{N_{2T}}$. Since $F_1(t;v)$ can be considered as a bilinear map (4.1), the corresponding estimate is stated as follows

**Lemma 4.4.** *Let $\omega_j(t)$ be states associated with correlation functions $(\Gamma_j(t), \Lambda_j(t))$ $j = 1, 2$, for any $T \in \mathbb{R}$,*

$$\|B_1(\omega_1(t), \omega_2(t); v)\|_{L_t^1([0,T],\mathcal{L}^1)} \lesssim \left(T^{\frac{\epsilon}{4}} + T\right) \|v\|_M \|\omega_1(t)\|_{N_T} \|\omega_2(t)\|_{N_T}. \tag{4.13}$$

*Proof.* The estimate (4.13) is the summary of the following results

(a) $\left\|\left[\rho_{\Gamma_1(t)} * v, \Gamma_2(t)\right]\right\|_{L_t^1([0,T],\mathcal{L}^1)} \lesssim \left(\|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}} T^{\frac{\epsilon}{4}} + \|v\chi_2\|_{L^\infty} T\right) \|\Gamma_1(t)\|_{N_{1T}} \|\Gamma_2(t)\|_{N_{1T}}$;

(b) $\|[\Gamma_1(t), \Gamma_2(t)]_v\|_{L_t^\infty([0,T],\mathcal{L}^1)} \lesssim (\|(v\chi_1)(x)|x|\|_{L^3} + \|v\chi_2\|_{L^\infty}) \|\Gamma_1(t)\|_{N_{1T}} \|\Gamma_2(t)\|_{N_{1T}}$;

(c) $\|[\Lambda_1(t), \Lambda_2^*(t)]_v\|_{L_t^\infty([0,T],\mathcal{L}^1)} \lesssim (\|(v\chi_1)(x)|x|\|_{L^3} + \|v\chi_2\|_{L^\infty}) \|\Lambda_1(t)\|_{N_{2T}} \|\Lambda_2(t)\|_{N_{2T}}$.

To show (a), using the operator inequality,

$$
\begin{aligned}
\left\|\left[\rho_{\Gamma_1(t)} * v, \Gamma_2(t)\right]\right\|_{L_t^1([0,T],\mathcal{L}^1)} &\leq 2\|\rho_{\Gamma_1(t)} * v\|_{L_t^1([0,T],op)} \|\Gamma_2(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)} \\
&\leq 2\|\rho_{\Gamma_1(t)} * v\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} \|\Gamma_2(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)}.
\end{aligned}
$$

Then estimate $\|\rho_{\Gamma_1(t)} * v\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)}$ by decomposing $v$ as $v\chi_1$ and $v\chi_2$. For $\rho_{\Gamma_1(t)} * (v\chi_2)$,

$$
\begin{aligned}
\left\|\rho_{\Gamma_1(t)} * (v\chi_2)\right\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} &\leq \|v\chi_2\|_{L^\infty} \|\rho_{\Gamma_1(t)}\|_{L_t^1 L_x^1([0,T]\times\mathbb{R}^3)} \\
&\leq \|v\chi_2\|_{L^\infty} T \|\Gamma_1(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)}.
\end{aligned}
$$

For $\rho_{\Gamma_1(t)} * (v\chi_1)$, by the Young's convolution inequality and the Hölder inequality

$$
\left\|\rho_{\Gamma_1(t)} * (v\chi_1)\right\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^3)} \leq \|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}} T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1(t)}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)}.
$$

Combining above two estimates by the triangle inequality, we obtain

$$
\begin{aligned}
&\left\|[\rho_{\Gamma_1(t)} * v, \Gamma_2(t)]\right\|_{L_t^1([0,T],\mathcal{L}^1)} \\
\lesssim &\|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}} T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1(t)}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)} \|\Gamma_2(t)\|_{L_t^\infty([0,T],\mathcal{L}^1)} \\
&+ \|v\chi_2\|_{L^\infty} T \|\Gamma_2(t)\|^2_{L_t^\infty([0,T],\mathcal{L}^1)},
\end{aligned}
$$

which implies (a).

As for estimates (b) and (c), we adopt a fixed time argument. Since they are treated in the same way, for simplicity, we only state the proof of (b). At a fixed time,

$$
\|(v\Gamma_1)\Gamma_2\|_{\mathcal{L}^1} = \left\|(v\Gamma_1)\langle\nabla\rangle^{-1}\langle\nabla\rangle\Gamma_2\right\|_{\mathcal{L}^1} \leq \left\|(v\Gamma_1)\langle\nabla\rangle^{-1}\right\|_{\mathcal{L}^2} \|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}.
$$

Similar as the proof of estimate (a), we decompose $v$ into $v\chi_1$ and $v\chi_2$. The part consisting of $v\chi_2$ is estimated trivially,

$$
\left\|((v\chi_2)\Gamma_1)\langle\nabla\rangle^{-1}\right\|_{\mathcal{L}^2} \|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2} \leq \|v\chi_2\|_{L^\infty} \|\Gamma_1(x,y)\|_{L^2_{x,y}} \|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}.
$$

Using the Hardy's inequality to deal with the part containing $v\chi_1$, we obtain

$$\begin{aligned}
&\left\|((v\chi_1)\Gamma_1)\langle\nabla\rangle^{-1}\right\|_{\mathcal{L}^2}\|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}\\
\lesssim&\|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}\|(v\chi_1)(x-y)|x-y|\Gamma_1(x,y)\|_{L^2_{x,y}}\quad\text{(Hardy's inequality)}\\
\leq&\|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}\|(v\chi_1)(x)|x|\|_{L^3}\|\Gamma_1(x,y)\|_{L^6_{x-y}L^2_{x+y}}\quad\text{(Hölder's inequality)}\\
\lesssim&\|\langle\nabla\rangle\Gamma_2\|_{\mathcal{L}^2}\|(v\chi_1)(x)|x|\|_{L^3}\|\langle\nabla_{x-y}\rangle\Gamma_1(x,y)\|_{L^2_{x-y}L^2_{x+y}}\quad\text{(Sobolev inequality)}.
\end{aligned}$$

□

Case II and III involve Strichartz norms defined in Definition 2.11, which are basically derived from estimates of the linear parts for Equation (2.39) and (2.40), where $v\Lambda$ is excluded from the linear part of Equation (2.40). In order to handle the singular term $v\Lambda$ in Equation (2.40), we do not include the endpoint case of the Strichartz norms for $\Lambda$. For the application to the local well-posed result Theorem 2.15, it suffices to put the forcing terms $F_1(t,x,y;v)$ and $F_2(t,x,y;v)$ in $L^1_tH^1$, and $\langle\nabla_{x,y}\rangle(v\Lambda)$ in the dual Strichartz space $L^2_tL^{6/5}_{x-y}L^2_{x+y}$. The involved estimates are summarized as

**Lemma 4.5.** *Let $\Gamma(t,x,y)$ and $\Lambda(t,x,y)$ be solutions to the linear equations*

$$\begin{cases} i\,\partial_t\Gamma(t,x,y)=(-\Delta_x+\Delta_y)\Gamma(t,x,y)\\ i\,\partial_t\Lambda(t,x,y)=(-\Delta_x-\Delta_y)\Lambda(t,x,y)\end{cases},\quad t\in\mathbb{R},\ x,y\in\mathbb{R}^3,$$

*with the initial data*

$$\Gamma(0,x,y) = \Gamma_0(x,y) \quad \textit{and} \quad \Lambda(0,x,y) = \Lambda_0(x,y),$$

*then for any $s \in \mathbb{R}$, the following Strichartz estimates hold*

$$\|\Gamma(t,x,y)\|_{ST^s_\infty} \lesssim \|\Gamma_0(x,y)\|_{H^s} \quad \textit{and} \quad \|\Lambda(t,x,y)\|_{ST^s_{\epsilon\infty}} \lesssim \|\Lambda_0(x,y)\|_{H^s}. \tag{4.14}$$

*Furthermore, if $\Gamma(t,x,y)$ and $\Lambda(t,x,y)$ are solutions to the inhomogeneous equations*

$$\begin{cases} i\,\partial_t\Gamma(t,x,y) = (-\Delta_x + \Delta_y)\Gamma(t,x,y) + F(t,x,y) \\ i\,\partial_t\Lambda(t,x,y) = (-\Delta_x - \Delta_y)\Lambda(t,x,y) + G(t,x,y) \end{cases},$$

*where $t \in \mathbb{R}, x, y \in \mathbb{R}^3$, then for any $s, T \in \mathbb{R}$,*

$$\|\Gamma(t,x,y)\|_{ST^s_T} \lesssim \|\Gamma_0(x,y)\|_{H^s} + \|\langle\nabla_{x,y}\rangle^s F(t,x,y)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}$$

*and*

$$\|\Lambda(t,x,y)\|_{ST^s_{\epsilon T}} \lesssim \|\Lambda_0(x,y)\|_{H^s} + \|\langle\nabla_{x,y}\rangle^s G(t,x,y)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}.$$

*or*

$$\|\Lambda(t,x,y)\|_{ST^s_{\epsilon T}} \lesssim \|\Lambda_0(x,y)\|_{H^s} + \|\langle\nabla_{x,y}\rangle^s F_2(t,x,y)\|_{L^2_t L^{6/5}_{x-y} L^2_{x+y}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}.$$

*Proof.* Since $\langle\nabla_x\rangle^s\langle\nabla_y\rangle^s$ commutes with operators in the linear equations, using formulas for the solutions of the linear equations and the Strichartz estimates [KT98], we obtain the estimates (4.14). Then the inhomogeneous estimates follow from the Christ-Kiselev lemma [Tao06, Lemma 2.4]. □

As an application of Lemma 4.5 to Equation (2.40) and Proposition 4.19 the Morrey's inequality for Banach spaces, we have

**Lemma 4.6.** *Consider the equation*

$$\begin{cases} i\,\partial_t \Lambda(t,x,y) = (-\Delta_x - \Delta_y)\Lambda(t,x,y) + v(x-y)\Lambda(t,x,y) + F(t,x,y) \\ \Lambda(t,y,x) = -\Lambda(t,x,y) \end{cases},$$

*where* $t \in \mathbb{R}, x, y \in \mathbb{R}^3$*, and the initial condition* $\Lambda(0,x,y) = \Lambda_0(x,y)$*, for sufficiently short time* $T$ *such that*

$$C \max\{T, T^{\epsilon/4}\} \|v\|_M \leq \frac{1}{2},$$

*where* $C$ *is a universal constant, then the solution* $\Lambda(t,x,y)$ *satisfies the estimate*

$$\|\Lambda(t,x,y)\|_{N_{2T}} \lesssim \|\Lambda_0(x,y)\|_{H^1} + \|F(t,x,y)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}. \tag{4.15}$$

*Proof.* Treat $v(x-y)\Lambda(t,x,y)$ as a forcing term and apply Lemma 4.5,

$$\begin{aligned} \|\Lambda(t,x,y)\|_{N_{2T}} \lesssim \|\Lambda_0(x,y)\|_{H^1} + \|\langle \nabla_{x,y}\rangle \left(v(x-y)\Lambda(t,x,y)\right)\|_{L^2_t L^{6/5}_{x-y} L^2_{x+y}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \\ + \|F(t,x,y)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}, \end{aligned}$$

and the goal is to absorb $\|\langle \nabla_{x,y}\rangle \left(v(x-y)\Lambda(t,x,y)\right)\|_{L^2_t L^{6/5}_{x-y} L^2_{x+y}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$ to the left hand side.

Note that $\|\langle\nabla_{x,y}\rangle\left(v(x-y)\Lambda(t,x,y)\right)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$ is majorized by

$$\begin{aligned}
&\|v(x-y)\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
&+\|\left(\nabla v(x-y)\right)\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
&+\|v(x-y)\nabla_x\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
&+\|v(x-y)\nabla_y\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)},
\end{aligned}$$

where $\|\left(\nabla v(x-y)\right)\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$ is the most singular term. For simplicity, we only state the argument for $\left(\nabla v(x-y)\right)\Lambda(t,x,y)$. Considering the decomposition of $v$ as $v\chi_1$ and $v\chi_2$, terms involving $v\chi_2$ are essentially bounded by

$$\|\langle\nabla\rangle(v\chi_2)\|_{L^3} T\|\Lambda(t,x,y)\|_{L_t^\infty H^1([0,T]\times\mathbb{R}^6)},$$

and the terms involving $v\chi_1$ can be estimated as follows,

$$\begin{aligned}
&\|\nabla(v\chi_1)(x-y)\Lambda(t,x,y)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
=&\left\|\nabla(v\chi_1)(x-y)|x-y|^{(1-\epsilon)/2}\frac{\|\Lambda(t,x,y)\|_{L_{x+y}^2}}{|x-y|^{(1-\epsilon)/2}}\right\|_{L_t^2 L_{x-y}^{6/5}([0,T]\times\mathbb{R}^3)}\\
\lesssim&\left\|\nabla(v\chi_1)(x-y)|x-y|^{(1-\epsilon)/2}\|\langle\nabla_{x-y}\rangle\Lambda(t,x,y)\|_{L_{x-y}^{6/(1+\epsilon)} L_{x+y}^2}\right\|_{L_t^2 L_{x-y}^{6/5}([0,T]\times\mathbb{R}^3)}\\
&\text{(by Proposition 4.19 and } \|\Lambda(t,x,x)\|_{L_x^2}=0)\\
\leq&\left\|\nabla(v\chi_1)|x|^{(1-\epsilon)/2}\right\|_{L^{6/5}} T^{\epsilon/4}\|\langle\nabla_{x-y}\rangle\Lambda(t,x,y)\|_{L_t^{4/(2-\epsilon)} L_{x-y}^{(6/(1+\epsilon))} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}\\
&\text{(Hölder inequality in t).}
\end{aligned}$$

Therefore, for sufficiently small $T$ such that $C\max\{T, T^{\epsilon/4}\}\|v\|_M \le 1/2$, where $C$ is essentially the constant shown in Morrey's inequality,

$$\|\langle\nabla_{x,y}\rangle\left(v(x-y)\Lambda(t,x,y)\right)\|_{L_t^2 L_{x-y}^{6/5} L_{x+y}^2([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$$

is absorbed to the left hand side. $\square$

According to Lemma 4.5 and 4.6 , for Case II and III, it remains to estimate $\|F_1(t,x,y;v)\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$ and $\|F_2(t,x,y;v)\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$ by $\|\Gamma(t)\|_{N_{1T}}$ and $\|\Lambda(t)\|_{N_{2T}}$. Similar to Lemma 4.4, the result is still stated in terms of corresponding bilinear maps $B_1$ and $B_2$.

**Lemma 4.7.** *Let $\omega_j(t)$ be states associated with correlation functions $(\Gamma_j(t), \Lambda_j(t))$ $j = 1, 2$, for any $T \in \mathbb{R}$,*

$$\|B_j\left(\omega_1(t), \omega_2(t); v\right)\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)} \lesssim \left(T^{\frac{\epsilon}{4}} + T\right)\|v\|_M \|\omega_1(t)\|_{N_T}\|\omega_2(t)\|_{N_T}, \quad j = 1, 2. \tag{4.16}$$

*Proof.* For simplicity, we omit the notation $t$ of $\Gamma_j(t)$ and $\Lambda_j(t)$, $j = 1, 2$ for the time being. The estimates (4.16) are the summaries of the following results

(a) Estimates involving $\rho_\Gamma * v$:

(a.1) $\|[\rho_{\Gamma_1} * v, \Gamma_2]\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$ and (a.2) $\|[\rho_{\Gamma_1} * v, \Lambda_2]_+\|_{L_t^1 H^1([0,T]\times\mathbb{R}^6)}$

$$\begin{aligned}&\lesssim \left(\left(\|\chi_1\nabla v\|_{L^{\frac{3}{3-\epsilon/2}}} + \|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}}\right)T^{\frac{\epsilon}{4}} + \left(\|\chi_2\nabla v\|_{L^\infty} + \|v\chi_2\|_{L^\infty}\right)T\right)\\ &\quad\cdot\|\Gamma_1\|_{N_{1T}}\|\Gamma_2\|_{N_{1T}}.\end{aligned}$$

(b) Estimates involving $v$:

(b.1) $\|[\Gamma_1,\Gamma_2]_v\|_{L_t^1H^1([0,T]\times\mathbb{R}^6)}$, (b.2) $\|[\Lambda_1,\Lambda_2^*]_v\|_{L_t^1H^1([0,T]\times\mathbb{R}^6)}$,

(b.3) $\|[\Gamma_1,\Lambda_2]_{v,+}\|_{L_t^1H^1([0,T]\times\mathbb{R}^6)}$ and $\left\|[\Lambda_1,\bar{\Gamma}_2]_{v,+}\right\|_{L_t^1H^1([0,T]\times\mathbb{R}^6)}$

$$\lesssim \left(\|v\chi_1\|_{L^{\frac{3}{2+\epsilon/2}}}T^{1/4}+\|v\chi_1\|_{L^{\frac{12}{8-\epsilon}}}T^{\frac{4+\epsilon}{8}}+\|v\chi_2\|_{L^\infty}T\right)\|\Gamma_1\|_{N_{1T}}\|\Gamma_2\|_{N_{1T}}.$$

Since the proof of (a.2) is essentially the same as (a.1), we demonstrate the argument for (a.1) only. Considering two typical terms

$$\|\nabla\circ(\rho_{\Gamma_1}*v)\circ\Gamma_2\|_{L_t^1([0,T],\mathcal{L}^2)}\quad\text{and}\quad\|\,(\rho_{\Gamma_1}*v)\,\Gamma_2\|_{L_t^1([0,T],\mathcal{L}^2)}$$

in (a.1), we have estimates

$$\begin{aligned}
&\|\nabla\circ(\rho_{\Gamma_1}*v)\circ\Gamma_2\|_{L_t^1([0,T],\mathcal{L}^2)}\\
&\le\|\nabla(\rho_{\Gamma_1}*v)\Gamma_2\|_{L_t^1L^2([0,T]\times\mathbb{R}^6)}+\|(\rho_{\Gamma_1}*v)(x)\nabla_x\Gamma_2(x,y)\|_{L_t^1L^2([0,T]\times\mathbb{R}^6)}\\
&\le\|\nabla(\rho_{\Gamma_1}*v)\Gamma_2\|_{L_t^1L^2([0,T]\times\mathbb{R}^6)}+\|\rho_{\Gamma_1}*v\|_{L_t^1L_x^\infty([0,T]\times\mathbb{R}^3)}\|\Gamma_2(x,y)\|_{L_t^\infty H^1([0,T]\times\mathbb{R}^6)},
\end{aligned}$$

and

$$\begin{aligned}
\|(\rho_{\Gamma_1}*v)\Gamma_2\|_{L_t^1([0,T],\mathcal{L}^2)}&\le\|\rho_{\Gamma_1}*v\|_{L_t^1([0,T],op)}\|\Gamma_2\|_{L_t^\infty([0,T],\mathcal{L}^2)}\\
&\le\|\rho_{\Gamma_1}*v\|_{L_t^1L_x^\infty([0,T]\times\mathbb{R}^3)}\|\Gamma_2\|_{L_t^\infty H^1([0,T]\times\mathbb{R}^6)},
\end{aligned}$$

where the potential $v$ is handled as in the proof of Lemma 4.4,

$$\begin{aligned}
&\|\rho_{\Gamma_1} * v\|_{L^1_t L^\infty_x([0,T]\times\mathbb{R}^3)} \\
\leq &\|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}} T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)} + \|v\chi_2\|_{L^\infty} T \|\Gamma_1\|_{L^\infty_t([0,T],\mathcal{L}^1)},
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla(\rho_{\Gamma_1} * v)\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)} \\
\leq &\|(\rho_{\Gamma_1} * (\chi_1 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)} + \|(\rho_{\Gamma_1} * (\chi_2 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}\,. \qquad (4.17)
\end{aligned}$$

In (4.17), we estimate $\|(\rho_{\Gamma_1} * (\chi_1 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}$ using the functional inequality,

$$\|(\rho_{\Gamma_1} * (\chi_2 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)} \leq \|\chi_2 \nabla v\|_{L^\infty} T \|\Gamma_1\|_{L^\infty_t([0,T],\mathcal{L}^1)} \|\Gamma_2\|_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)},$$

and estimate $\|(\rho_{\Gamma_1} * (\chi_1 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}$ as follows,

$$\begin{aligned}
&\|(\rho_{\Gamma_1} * (\chi_1 \nabla v))\,\Gamma_2\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)} \\
&\leq \|\rho_{\Gamma_1} * (\chi_1 \nabla v)\|_{L^1_t L^3_x([0,T]\times\mathbb{R}^3)} \|\Gamma_2(x,y)\|_{L^\infty_t L^6_x L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \\
&\quad \text{(Hölder inequality in x and t)} \\
&\lesssim \|\chi_1 \nabla v\|_{L^{\frac{3}{3-\epsilon/2}}} T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)} \|\langle\nabla_x\rangle\Gamma(x,y)\|_{L^\infty_t L^2([0,T]\times\mathbb{R}^6)} \\
&\quad \text{(by Young's inequality and Sobolev inequality)} \\
&\leq \|\chi_1 \nabla v\|_{L^{\frac{3}{3-\epsilon/2}}} T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)} \|\Gamma_2\|_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)}.
\end{aligned}$$

Assembling above estimates,

$$\begin{aligned}
&\|[\rho_{\Gamma_1} * v, \Gamma_2]\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)} \\
\lesssim &\Big(\|\chi_1 \nabla v\|_{L^{\frac{3}{3-\epsilon/2}}} + \|v\chi_1\|_{L^{\frac{3}{2-\epsilon/2}}}\Big) T^{\frac{\epsilon}{4}} \|\rho_{\Gamma_1}\|_{L_t^{\frac{1}{1-\epsilon/4}} L_x^{\frac{3}{1+\epsilon/2}}([0,T]\times\mathbb{R}^3)} \|\Gamma_2\|_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)} \\
&+ (\|\chi_2 \nabla v\|_{L^\infty} + \|v\chi_2\|_{L^\infty})\, T \|\Gamma_1\|_{L^\infty_t([0,T],\mathcal{L}^1)} \|\Gamma_2\|_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)}
\end{aligned}$$

which implies (a.1).

Considering all terms in group (b), they share similar structures and can be handled in the same method. For simplicity, we only show the proof for

$$\|(v\Gamma_1)\Gamma_2\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}$$

in (b.1). Note that

$$\begin{aligned}
&\|(v\Gamma_1)\Gamma_2\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)} \\
&\sim \|(v\Gamma_1)\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^2)} + \|\nabla\circ(v\Gamma_1)\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^2)} + \|(v\Gamma_1)\Gamma_2\circ\nabla\|_{L^1_t([0,T],\mathcal{L}^2)}
\end{aligned}$$

and $\|(v\Gamma_1)\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^2)}$ is majorized by $\|(v\Gamma_1)\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^1)}$, whose estimate is shown in Lemma 4.4. It remains to estimate $\nabla\circ(v\Gamma_1)\Gamma_2$ and $(v\Gamma_1)\Gamma_2\circ\nabla$.

Based on the observation that for an operator k,

$$\begin{aligned}
[\nabla, vk] &= \nabla\,(vk) - (vk)\,\nabla \\
&= \nabla_x\,(v(x-y)k(x,y)) + \nabla_y\,(v(x-y)k(x,y)) \\
&= v(x-y)\,(\nabla_x k(x,y) + \nabla_y k(x,y))\,,
\end{aligned}$$

the estimate for $\nabla(v\Gamma_1)\Gamma_2$ reduces to

$$\begin{aligned}
&\|(v\Gamma_1)\nabla\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^2)} \\
&+\left\|\int_{\mathbb{R}^3} dz\,\left(v(x-z)\,(\nabla_x\Gamma_1(x,z)+\nabla_z\Gamma_1(x,z))\right)\Gamma_2(z,y)\right\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}. \qquad (4.18)
\end{aligned}$$

Next decompose the potential $v$ as $v\chi_1$ and $v\chi_2$. By the triangle inequality, there are four terms in (4.18) to estimate. The terms involving $v\chi_2$ are relatively easier to handle and we have estimates

$$\|(v\chi_2\Gamma_1)\nabla\Gamma_2\|_{L^1_t([0,T],\mathcal{L}^2)} \le \|v\chi_2\|_{L^\infty} T\|\Gamma_1\|^2_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)}\|\Gamma_2\|^2_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)}$$

and

$$\left\| \int_{\mathbb{R}^3} dz\, (v\chi_2)(x-z) \nabla_x \Gamma_1(x,z) \Gamma_2(z,y) \right\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}$$
$$\leq \|v\chi_2\|_{L^\infty} T \|\Gamma_1\|^2_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)} \|\Gamma_2\|^2_{L^\infty_t H^1([0,T]\times\mathbb{R}^6)}.$$

For the other two terms involving $v\chi_1$,

$$\left\| \int_{\mathbb{R}^3} dz\, (v\chi_1)(x-z) \Gamma_1(x,z) \nabla_z \Gamma_2(z,y) \right\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)}$$

$$\leq \left\| \int_{\mathbb{R}^3} dz\, |v\chi_1|(z) \|\Gamma_1(x,x-z)\nabla_z\Gamma_2(x-z,y)\|_{L^2_x L^2_y} \right\|_{L^1_t([0,T])}$$

(Minkowski inequality)

$$\leq \left\| \int_{\mathbb{R}^3} dz\, |v\chi_1|(z) \|\Gamma_1(x,x-z)\|_{L_x^{\frac{3}{1-\epsilon/2}}} \|\nabla_z\Gamma_2(x-z,y)\|_{L_x^{\frac{6}{1+\epsilon}} L^2_y} \right\|_{L^1_t([0,T])}$$

(Hölder inequality in x)

$$\leq \|v\chi_1\|_{L^{\frac{3}{2+\epsilon/2}}} T^{1/4} \|\Gamma_1(x,x-z)\|_{L_t^{\frac{4}{1+\epsilon}} L_z^{\frac{3}{1-\epsilon/2}} L_x^{\frac{3}{1-\epsilon/2}}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \|\nabla_x\Gamma_2(x,y)\|_{L_t^{\frac{4}{2-\epsilon}} L_x^{\frac{6}{1+\epsilon}} L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$$

(Hölder inequality in z and t)

$$= \|v\chi_1\|_{L^{\frac{3}{2+\epsilon/2}}} T^{1/4} \|\Gamma_1(x,z)\|_{L_t^{\frac{4}{1+\epsilon}} L_x^{\frac{3}{1-\epsilon/2}} L_z^{\frac{3}{1-\epsilon/2}}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \|\nabla_x\Gamma_2(x,y)\|_{L_t^{\frac{4}{2-\epsilon}} L_x^{\frac{6}{1+\epsilon}} L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$$

(by Fubini's theorem)

$$\lesssim \|v\chi_1\|_{L^{\frac{3}{2+\epsilon/2}}} T^{1/4} \|\langle\nabla_z\rangle\Gamma_1(x,z)\|_{L_t^{\frac{4}{1+\epsilon}} L_x^{\frac{3}{1-\epsilon/2}} L^2_z([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \|\nabla_x\Gamma_2(x,y)\|_{L_t^{\frac{4}{2-\epsilon}} L_x^{\frac{6}{1+\epsilon}} L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)}$$

(Sobolev inequality),

and similarly,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^3} dz\, (v\chi_1)(x-z) \nabla_x \Gamma_1(x,z) \Gamma_2(z,y) \right\|_{L^1_t L^2([0,T]\times\mathbb{R}^6)} \\
\le &\left\| \int_{\mathbb{R}^3} dz\, |v\chi_1|(z) \| \nabla_x \Gamma_1(x,x-z) \Gamma_2(x-z,y) \|_{L^2_{x,y}} \right\|_{L^1_t([0,T])} \\
&\quad \text{(Minkowski inequality)} \\
\le &\left\| \int_{\mathbb{R}^3} dz\, |v\chi_1|(z) \| \nabla_x \Gamma_1(x,x-z) \|_{L_x^{\frac{12}{4+\epsilon}}} \| \Gamma_2(x-z,y) \|_{L_x^{\frac{12}{2-\epsilon}} L^2_y} \right\|_{L^1_t([0,T])} \\
&\quad \text{(Hölder inequality in x)} \\
\lesssim &\left\| \int_{\mathbb{R}^3} dz\, |v\chi_1|(z) \| \nabla_x \Gamma_1(x,x-z) \|_{L_x^{\frac{12}{4+\epsilon}}} \| \nabla_x \Gamma_2(x,y) \|_{L_x^{\frac{12}{6-\epsilon}} L^2_y} \right\|_{L^1_{[0,T]}} \\
&\quad \text{(Sobolev inequality)} \\
\le &\| v\chi_1 \|_{L^{\frac{12}{8-\epsilon}}} T^{\frac{4+\epsilon}{8}} \| \nabla_x \Gamma_1(x,x-z) \|_{L_t^{\frac{4}{2-\epsilon}} L_z^{\frac{12}{4+\epsilon}} L_x^{\frac{12}{4+\epsilon}}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \| \nabla_x \Gamma_2(x,y) \|_{L_t^{\frac{8}{\epsilon}} L_x^{\frac{12}{6-\epsilon}} L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \\
&\quad \text{(Hölder inequality in z and t)} \\
\le &\| v\chi_1 \|_{L^{\frac{12}{8-\epsilon}}} T^{\frac{4+\epsilon}{8}} \left( \| \nabla_x \Gamma_1(x,z) \|_{L_t^{\frac{4}{2-\epsilon}} L^{\frac{12}{4+\epsilon}}([0,T]\times\mathbb{R}^6)} + \| \nabla_z \Gamma_1(x,z) \|_{L_t^{\frac{4}{2-\epsilon}} L^{\frac{12}{4+\epsilon}}([0,T]\times\mathbb{R}^6)} \right) \\
&\cdot \| \nabla_x \Gamma_2(x,y) \|_{L_t^{\frac{8}{\epsilon}} L_x^{\frac{12}{6-\epsilon}} L^2_y([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \quad \text{(Fubini's theorem).}
\end{aligned}$$

□

With all the ingredients, the proof of the local well-posedness Theorem 2.15 is as follows

*Proof.* Given $\Gamma_0^* = \Gamma_0$ and $\Lambda_0^* = -\bar{\Lambda}$, if $\tilde{\Gamma}^*(t) = \tilde{\Gamma}(t)$ and $\tilde{\Lambda}^*(t) = -\bar{\tilde{\Lambda}}(t)$, after applying the Duhamel's formulation (2.51) to $\left(\tilde{\Gamma}(t), \tilde{\Lambda}(t)\right)$, the result $(\Gamma(t), \Lambda(t))$ still satisfies $\Gamma^*(t) = \Gamma(t)$ and $\Lambda^*(t) = -\bar{\Lambda}(t)$, and $F_1^*(t;v) = -F_1(t;v)$. By Lemma 4.3, 4.5 and

4.6, for sufficiently short time $T$, we have that

$$\|\Gamma(t)\|_{N_{1T}} + \|\Lambda(t)\|_{N_{2T}} \lesssim \|\Gamma_0\|_{\mathcal{L}^1} + \|F_1(t;v)\|_{L^1_{[0,T]}\mathcal{L}^1} + \|F_1(t,x,y;v)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)} + \|F_2(t,x,y;v)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)}.$$

Then employ Lemma 4.4 and 4.7,

$$\|F_1(t;v)\|_{L^1_{[0,T]}\mathcal{L}^1} \lesssim \left(T^{\frac{\epsilon}{4}} + T\right)\|v\|_M \left(\|\tilde{\Gamma}(t)\|_{N_{1T}} + \|\tilde{\Lambda}(t)\|_{N_{2T}}\right)^2$$

$$\|F_j(t,x,y;v)\|_{L^1_t H^1([0,T]\times\mathbb{R}^6)} \lesssim \left(T^{\frac{\epsilon}{4}} + T\right)\|v\|_M \left(\|\tilde{\Gamma}(t)\|_{N_{1T}} + \|\tilde{\Lambda}(t)\|_{N_{2T}}\right)^2, \quad j = 1, 2.$$

If necessary, choose a smaller $T$ such that

$$\left(T^{\frac{\epsilon}{4}} + T\right)\|v\|_M \left(\|\tilde{\Gamma}(t)\|_{N_{1T}} + \|\tilde{\Lambda}(t)\|_{N_{2T}}\right)^2$$

is small enough and the local well-posedness result follows from the standard Banach fixed point argument. □

## 4.4 Smooth Potential Case

Given any smooth initial data, if we can always obtain a smooth solution of Bogoliubov-de Gennes equations (2.43) and (2.44), it is straightforward to show the conservation of trace of and the conservation of energy. However when the potential $v$ is not smooth, due to the singular term $v(x-y)\Lambda(t,x,y)$ of Equation (2.44), the high regularity of initial data may not be preserved by Equation (2.43) and (2.44).

Therefore in this section, we assume the potential $\tilde{v}$ in the Bogoliubov-de Gennes equations is smooth, i.e.

$$\tilde{v} \in C_c^\infty(\mathbb{R}^3), \quad \tilde{v}(x) = \tilde{v}(-x) \quad \text{and} \quad v(x) \in \mathbb{R} \quad \text{for } x \in \mathbb{R}^3,$$

and recall that the equations are

$$i\,\partial_t \Gamma(t) = [-\Delta, \Gamma(t)] + \underbrace{[\tilde{v} * \rho_{\Gamma(t)}, \Gamma(t)] - [\Gamma(t), \Gamma(t)]_{\tilde{v}} + [\Lambda(t), \Lambda^*(t)]_{\tilde{v}}}_{F_1(t;\tilde{v})}, \tag{4.19}$$

$$i\,\partial_t \Lambda(t) = [-\Delta, \Lambda(t)]_+ + (\tilde{v}\Lambda)(t) + \underbrace{[\tilde{v} * \rho_{\Gamma(t)}, \Lambda(t)]_+ - [\Gamma(t), \Lambda(t)]_{\tilde{v},+} - [\Lambda(t), \bar{\Gamma}(t)]_{\tilde{v},+}}_{F_2(t;\tilde{v})}. \tag{4.20}$$

The corresponding integral equations are

$$\Gamma(t) = e^{i\Delta t}\Gamma_0 e^{-i\Delta t} - i\int_0^t ds\, e^{i\Delta(t-s)} F_1(s;\tilde{v})\, e^{-i\Delta(t-s)} \tag{4.21}$$

$$\Lambda(t) = e^{i\Delta t}\Lambda_0 e^{i\Delta t} - i\int_0^t ds\, e^{i\Delta(t-s)} \left((\tilde{v}\Lambda)(s) + F_2(s;\tilde{v})\right) e^{i\Delta(t-s)}. \tag{4.22}$$

In the smooth potential case, we are able to prove the regularity of initial data is preserved by Equation (4.19) and (4.20), and the conservation of trace and energy by using smooth solutions.

The outline of our proofs is as follows: we first establish the local well-posedness result Proposition 4.9 of Equation (4.19) and (4.20), and show that the quasi-free conditions of the initial data are preserved along the evolution (Lemma 4.12). Based on Proposition 4.9 and Lemma 4.12, using a Grönwall argument, it follows that the existence time of local solutions depend on the trace norm of $\Gamma(t)$: as long as $\|\Gamma(t)\|_{\mathcal{L}^1}$ is finite, Equation (4.19) and (4.20) with initial data in $\mathcal{S}_{ad}$ are well-posed. While the conservation of trace basically follows from applying the cyclicity property of the trace functional to the Duhamel's formulation. Besides if $(\Gamma(t), \Lambda(t)) \in \mathcal{S}_{ad}$ , $\Gamma(t)$ is self-adjoint and positive , then $\mathrm{Tr}(\Gamma(t)) = \|\Gamma(t)\|_{\mathcal{L}^1}$ and we can extend our local solutions globally. In the end, we use the smooth solution $(\Gamma(t), \Lambda(t))$ of Equation (4.19) and (4.20) and compute explicitly the time derivative of the energy functional. The time derivative vanishes identically. Therefore the energy is preserved.

The local theory (Proposition 4.9) of Equation (4.19) and (4.20) is established as an application of the standard Banach fixed point argument to the Duhamel's formulation (4.21) and (4.22) together with the auxiliary Lemma 4.8.

**Lemma 4.8.** *Let $\omega_j$ be states associated with correlation functions $(\Gamma_j, \Lambda_j), j = 1, 2$, for any $s \geq 0$,*

$$\|B_j(\omega_1, \omega_2; \tilde{v})\|_{\mathcal{L}^{s,1}} \lesssim \|\tilde{v}\|_{W^{s,\infty}} \left(\|\Gamma_1\|_{\mathcal{L}^{s,1}} + \|\Lambda_1\|_{H^s}\right)\left(\|\Gamma_2\|_{\mathcal{L}^{s,1}} + \|\Lambda_2\|_{H^s}\right), \quad j = 1, 2.$$

*For any $\Lambda \in H^s(\mathbb{R}^6)$,*

$$\|\tilde{v}\Lambda\|_{H^s} \lesssim \|\langle\nabla\rangle^s \tilde{v}\|_{L^\infty} \|\Lambda\|_{H^s}.$$

*Proof.* For simplicity, we only argue for the terms $\tilde{v} * \rho_{\Gamma_1}\Gamma_2$ and $(\tilde{v}\Lambda_1)\Lambda_2^*$. Other terms in the bilinear maps $B_1$ and $B_2$ can be handled similarly. Using the operator inequality, the trace theorem and the Hölder inequality for Schatten norms,

$$\begin{aligned}
\|\langle\nabla\rangle^s (\tilde{v} * \rho_{\Gamma_1}\Gamma_2) \langle\nabla\rangle^s\|_{\mathcal{L}^1} &\leq \|\langle\nabla\rangle^s (\tilde{v} * \rho_{\Gamma_1}) \langle\nabla\rangle^{-s}\|_{op} \|\langle\nabla\rangle^s\Gamma_2\langle\nabla\rangle^s\|_{\mathcal{L}^1} \\
&\leq \|\langle\nabla\rangle^s (\tilde{v} * \rho_{\Gamma_1})\|_{L^\infty} \|\langle\nabla\rangle^s\Gamma_2\langle\nabla\rangle^s\|_{\mathcal{L}^1} \\
&\leq \|\langle\nabla\rangle^s\tilde{v}\|_{L^\infty}\|\rho_{\Gamma_1}\|_{L^1} \|\langle\nabla\rangle^s\Gamma_2\langle\nabla\rangle^s\|_{\mathcal{L}^1} \\
&\leq \|\langle\nabla\rangle^s\tilde{v}\|_{L^\infty}\|\Gamma_1\|_{\mathcal{L}^1} \|\langle\nabla\rangle^s\Gamma_2\langle\nabla\rangle^s\|_{\mathcal{L}^1}.
\end{aligned}$$

and

$$\begin{aligned}
\|\langle\nabla\rangle^s ((\tilde{v}\Lambda_1)\Lambda_2^*) \langle\nabla\rangle^s\|_{\mathcal{L}^1} &\leq \|\langle\nabla\rangle^s(\tilde{v}\Lambda_1)\|_{\mathcal{L}^2}\|\Lambda_2^*\langle\nabla\rangle^s\|_{\mathcal{L}^2} \\
&\leq \|\tilde{v}\|_{W^{s,\infty}}\|\Lambda_1\|_{H^s}\|\Lambda_2\|_{H^s} \quad \text{(by the fractional Leibniz rule).}
\end{aligned}$$

□

Notice that $\|B_2(\omega_1,\omega_2;\tilde{v})\|_{\mathcal{L}^{s,1}}$ majorizes $\|B_2(\omega_1,\omega_2;\tilde{v})\|_{H^s}$, which needs to be controlled when applying the Banach fixed point argument to Equation (4.20).

**Proposition 4.9.** *Let $s \geq 0$ and assume the initial data of the Bogoliubov-de Gennes equations (4.19) and (4.20) satisfy*

$$\Gamma_0 \in \mathcal{L}^{s,1} \quad \textit{and} \quad \Lambda_0 \in H^s.$$

*For sufficiently small time $T$, there is a unique mild solution $(\Gamma(t),\Lambda(t))$ of (4.19)* *and (4.20) such that*

$$\Gamma(t) \in C\left([0,T], \mathcal{L}^{s,1}\right), \quad \Lambda(t) \in C\left([0,T], H^s\right).$$

*Furthermore, if* $s \geq 2$,

$$\Gamma(t) \in C\left([0,T], \mathcal{L}^{s,1}\right) \cap C^1\left([0,T], \mathcal{L}^{s-2,1}\right), \quad \Lambda(t) \in C\left([0,T], H^s\right) \cap C^1\left([0,T], H^{s-2}\right).$$

The space $\mathcal{S}_{ad}$ has a nice structure and any matrix in $\mathcal{S}_{ad}$ can be approximated by smooth matrices

**Lemma 4.10.** *Let*

$$S_\omega = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & 1 - \bar{\Gamma} \end{pmatrix} \in \mathcal{S}_{ad},$$

*then there is a sequence of modified quasi-free states* $\omega_n$ *with*

$$S_{\omega_n} = \begin{pmatrix} P_n \Gamma P_n & P_n \Lambda P_n \\ P_n \Lambda^* P_n & 1 - P_n \bar{\Gamma} P_n \end{pmatrix}$$

*where* $P_n = 1_{|\nabla| \leq n}$ *is the truncation of frequency, converging to* $\omega$ *in the sense* $S_{\omega_n} \to S_\omega$ *in the strong operator topology. If* $\Gamma \in \mathcal{L}^{s,1}$, $\Lambda \in H^s$ *and* $s \geq 0$, $P_n \Gamma P_n \xrightarrow{\mathcal{L}^{s,1}} \Gamma$ *and* $P_n \Lambda P_n \xrightarrow{H^s} \Lambda$.

*Proof.* To verify that $S_{\omega_n}$ satisfies Condition (2.37), since $\bar{P}_n = P_n$ and $P_n^* = P_n$, it is straightforward to check $S_{\omega_n} + \mathcal{J} S_{\omega_n} \mathcal{J} = 1$ and $S_{\omega_n}^* = S_{\omega_n}$. As for $0 \leq S_{\omega_n} \leq 1$, choosing $f, g \in L^2(\mathbb{R}^3)$,

$$\left\langle S_{\omega_n} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \left\langle S_\omega \begin{pmatrix} P_n f \\ P_n g \end{pmatrix}, \begin{pmatrix} P_n f \\ P_n g \end{pmatrix} \right\rangle + \langle (1 - P_n) g, (1 - P_n) g \rangle \leq \|P_n f\|_{L^2} + \|g\|_{L^2}.$$

The convergence follows from the property $P_n f \xrightarrow{L^2} f,\ f \in L^2(\mathbb{R}^3)$. □

*Remark* 4.11. In the subspace of $\mathcal{S}_{ad}$ where $S^2 = S$, one can approximate $S$ using matrices within the subspace. However the approximation is not linear and depends on the Pin group representation (Proposition 6.20).

For any

$$S_\omega = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & 1 - \bar{\Gamma} \end{pmatrix} \in \mathcal{S}_{ad},$$

the functional inequality $1 \geq S_\omega \geq 0$ is equivalent to $S_\omega - S_\omega^2 \geq 0$, which implies

$$\Gamma - \Gamma^2 - \Lambda\Lambda^* \geq 0. \tag{4.23}$$

Following the same idea as [BSS18, Section 5.7.1.], if $\Gamma(t) \in \mathcal{L}^{2,1}$ and $\Lambda(t) \in H^2$, along Equation (4.19) and (4.20), the spectrum of the generalized one particle matrix $S_{\omega(t)}$ is preserved, where the state $\omega(t)$ is associated to $(\Gamma(t), \Lambda(t))$. Therefore if the initial state $\omega_0$ is quasi-free, then $\omega(t)$ remains quasi-free as long as Equation (4.19) and (4.20) are well-posed. The statement is summarized in Lemma 4.12 and the proof is given in the appendix.

**Lemma 4.12.** *Let $(\Gamma(t), \Lambda(t))$ be the solution to the Bogoliubov-de Gennes equa-*

*tions (4.19) and (4.20) such that*

$$\Gamma(t) \in C\left([0,T],\mathcal{L}^{2,1}\right) \cap C^1\left([0,T],\mathcal{L}^1\right), \quad \Lambda(t) \in C\left([0,T],H^2\right) \cap C^1\left([0,T],L^2\right),$$

*where $T > 0$. The spectrum of*

$$S_{\omega(t)} = \begin{pmatrix} \Gamma(t) & \Lambda(t) \\ \Lambda^*(t) & 1-\bar{\Gamma}(t) \end{pmatrix}$$

*does not change on $t \in [0,T]$. Furthermore if $(\Gamma(0),\Lambda(0)) \in \mathcal{S}_{ad}$, $(\Gamma(t),\Lambda(t)) \in \mathcal{S}_{ad}$ for $t \in [0,T]$.*

Once $(\Gamma,\Lambda) \in \mathcal{S}_{ad}$, $\Gamma$ must be a non-negative operator and the trace norm of $\Gamma$ is related to the Hilbert-Schmidt norms of $\Gamma$ and $\Lambda$ as shown in Lemma 4.13. The fact is crucial in our global theory.

**Lemma 4.13.** *Suppose $(\Gamma,\Lambda)$ satisfies (4.23), $\Gamma^* = \Gamma$ and $\Lambda^* = -\bar{\Lambda}$, then*

$$\|\Lambda\|_{H^s}^2 + \|\Gamma\|_{H^s}^2 \lesssim \|\Gamma\|_{\mathcal{L}^{s,1}}.$$

*Proof.* Let $\Lambda \in H^s$, $\Gamma \in \mathcal{L}^{s,1}$, according to the functional inequality (4.23), we have the following functional inequality on $L^2(\mathbb{R}^3)$,

$$\langle\nabla\rangle^s\Gamma\langle\nabla\rangle^s - (\langle\nabla\rangle^s\Gamma)\circ(\langle\nabla\rangle^s\Gamma)^* - (\langle\nabla\rangle^s\Lambda)\circ(\langle\nabla\rangle^s\Lambda)^* \geq 0.$$

Besides,

$$\begin{cases} \Gamma^* = \Gamma \\ \Lambda^* = -\bar{\Lambda} \end{cases} \implies \|\Gamma\|_{H^s} \sim \|\langle\nabla\rangle^s \Gamma\|_{\mathcal{L}^2}\,, \text{ and } \|\Lambda\|_{H^s} \sim \|\langle\nabla\rangle^s \Lambda\|_{\mathcal{L}^2}\,,$$

we obtain the desired result. □

As shown in Lemma 4.13, if a solution $(\Gamma(t), \Lambda(t))$ of the Bogoliubov-de Gennes equations (4.19) and (4.20) is quasi-free, to study how $\|\Gamma(t)\|_{\mathcal{L}^{s,1}}$ and $\|\Lambda(t)\|_{H^s}$ grow in time, it suffices to consider $\|\Gamma(t)\|_{\mathcal{L}^{s,1}}$ only. By a Grönwall argument, it further reduces to the problem of studying the growth of $\|\Gamma(t)\|_{\mathcal{L}^1}$.

**Proposition 4.14.** *Let $s \geq 0$ and for $t \in [0,T]$, $(\Gamma(t), \Lambda(t)) \in \mathcal{S}_{ad}$ be a solution to the integral equations (4.21) and (4.22), then*

$$\|\Gamma(t)\|_{\mathcal{L}^{s,1}} \leq \|\Gamma_0\|_{\mathcal{L}^{s,1}} \exp\left(C_s \|\tilde{v}\|_{W^{s,\infty}} (1 + \|\Gamma(t)\|_{\mathcal{L}^1})\, t\right).$$

*where $C_s$ is a constant depending on $s$.*

*Proof.* Applying the Minkowski inequality and the fact that $e^{i\Delta t}$ is a unitary operator to (4.21)

$$\|\Gamma(t)\|_{\mathcal{L}^{s,1}} \leq \|\Gamma_0\|_{\mathcal{L}^{s,1}} + \int_0^t d\tau\ \|F_1(\tau;\tilde{v})\|_{\mathcal{L}^{s,1}}\,. \tag{4.24}$$

To obtain a fixed time estimate for $F_1(t;\tilde{v})$, using the same proof of Lemma 4.8, we have

$$\left\|(\tilde{v}\Lambda)(t)\bar{\Lambda}(t)\right\|_{\mathcal{L}^{s,1}} \lesssim \|\tilde{v}\|_{W^{s,\infty}} \|\Lambda(t)\|_{H^s}^2,$$

$$\|(\tilde{v}\Gamma)(t)\Gamma(t)\|_{\mathcal{L}^{s,1}} \lesssim \|\tilde{v}\|_{W^{s,\infty}} \|\Gamma(t)\|_{H^s}^2,$$

$$\|(v * \rho_{\Gamma(t)})\Gamma(t)\|_{\mathcal{L}^{s,1}} \leq \|\tilde{v}\|_{W^{s,\infty}} \|\Gamma(t)\|_{\mathcal{L}^1} \|\Gamma(t)\|_{\mathcal{L}^{s,1}}.$$

Majorizing $\|\Lambda(t)\|_{H^s}^2$ and $\|\Lambda(t)\|_{H^s}^2$ by $\|\Gamma(t)\|_{\mathcal{L}^{s,1}}$ (Lemma 4.13), we obtain the estimate for $F_1(t;\tilde{v})$,

$$\|F_1(t;\tilde{v})\|_{\mathcal{L}^{s,1}} \leq C_s \|\tilde{v}\|_{W^{s,\infty}} \left(1 + \|\Gamma(t)\|_{\mathcal{L}^1}\right) \|\Gamma(t)\|_{\mathcal{L}^{s,1}},$$

and apply it to (4.24)

$$\|\Gamma(t)\|_{\mathcal{L}^{s,1}} \leq \|\Gamma_0\|_{\mathcal{L}^{s,1}} + \int_0^t d\tau\, C_s \|\tilde{v}\|_{W^{s,\infty}} \left(1 + \|\Gamma(\tau)\|_{\mathcal{L}^1}\right) \|\Gamma(\tau)\|_{\mathcal{L}^{s,1}}.$$

Then the result is an application of the Grönwall's inequality. □

The trace of $\Gamma(t)$ is preserved, which follows from the application of the cyclicity property to the Duhamel's formulation.

**Proposition 4.15.** *Suppose for $t \in [0,T]$, $(\Gamma(t), \Lambda(t))$ is a solution to the integral equations (4.21) and (4.22) and*

$$\Gamma(t) \in C\left([0,T], \mathcal{L}^1\right), \quad \Lambda(t) \in C\left([0,T], L^2\right),$$

*then the trace of* $\Gamma(t)$ *does not change on* $t \in [0,T]$.

*Proof.* At any fixed time $t \in [0,T]$,

$$\begin{aligned}\operatorname{Tr}(\Gamma(t)) &= \operatorname{Tr}\left(e^{i\Delta t}\Gamma_0 e^{-i\Delta t} - i\int_0^t ds\, e^{i\Delta(t-s)} F_1(s,\tilde{v})\, e^{-i\Delta(t-s)}\right) \\ &= \operatorname{Tr}(\Gamma_0) - i\int_0^t ds\, \operatorname{Tr}(F_1(s,\tilde{v})) \quad \text{(cyclicity of trace)},\end{aligned}$$

where

$$\operatorname{Tr}\left([\tilde{v} * \rho_{\Gamma(t)} - \tilde{v}\Gamma(t), \Gamma(t)]\right) = 0, \quad \text{(cyclicity of trace)}$$

and

$$\begin{aligned}&\operatorname{Tr}([\Lambda(t), \Lambda^*(t)]_{\tilde{v}}) \\ =&\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dz\, (\tilde{v}(x-z)\Lambda(t,x,z)\Lambda^*(t,z,x) - \tilde{v}(z-x)\Lambda(t,x,z)\Lambda^*(t,z,x)) \\ =&0.\end{aligned}$$

Therefore for any $t \in [0,T]$, $\operatorname{Tr}(\Gamma(t)) = \operatorname{Tr}(\Gamma_0)$. ☐

Combing all the above results, if we assume the initial data $(\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$ and sufficient regularity of $\Gamma_0$ and $\Lambda_0$, $\|\Gamma(t)\|_{\mathcal{L}^{s,1}}$ does not blow up at finite time. Therefore the solution $(\Gamma(t), \Lambda(t))$ to Equation (4.19) and (4.20) is global. The exact statement is

**Proposition 4.16.** *Consider the Bogoliubov-de Gennes equations (4.19) and (4.20) with the initial conditions* $\Gamma(t=0) = \Gamma_0$ *and* $\Lambda(t=0) = \Lambda_0$*, where* $(\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$*. Let* $s \geq 2$*, for arbitrary finite time* $T \in \mathbb{R}$*, there is a global mild solution* $(\Gamma(t), \Lambda(t))$

*existing on $[0,T]$ such that it satisfies all following properties*

*(i)* $\Gamma(t) \in C\left([0,T], \mathcal{L}^{s,1}\right) \cap C^1\left([0,T], \mathcal{L}^{s-2,1}\right)$,

*(ii)* $\Lambda(t) \in C\left([0,T], H^s\right) \cap C^1\left([0,T], H^{s-2}\right)$,

*(iii)* $(\Gamma(t), \Lambda(t)) \in \mathcal{S}_{ad}$ *for* $t \in [0,T]$,

*(iv)* $Tr(\Gamma(t)) = Tr(\Gamma_0)$ *for* $t \in [0,T]$.

Next we establish the conservation law of energy.

**Proposition 4.17.** *Let* $t \in [0,T]$, $T \in \mathbb{R}$, $\Gamma^*(t) = \Gamma(t)$ *and* $\Lambda^*(t) = -\bar{\Lambda}(t)$,

$\Gamma(t) \in C\left([0,T], \mathcal{L}^{4,1}\right) \cap C^1\left([0,T], \mathcal{L}^{2,1}\right)$ *and* $\Lambda(t) \in C\left([0,T], H^4\right) \cap C^1\left([0,T], H^2\right)$,

*and* $\Gamma(t)$ *and* $\Lambda(t)$ *satisfy the Bogoliubov-de Gennes equations (4.19) and (4.20), then*

$$\mathcal{E}_{BG}(\omega(t); \tilde{v}) = \mathcal{E}_{BG}(\omega(0); \tilde{v}), \quad t \in [0,T].$$

*Proof.* Differentiating the energy functional $\mathcal{E}_{BG}(\omega(t); \tilde{v})$ with respect to time $t$, for simplicity, we omit the notation $t$ and the result is the constant $-i$ times the sum of the following expressions, which are arranged in three groups

(a) $\mathrm{Tr}(-\Delta[-\Delta, \Gamma]) + \mathrm{Tr}((\rho_\Gamma * \tilde{v})[\rho_\Gamma * \tilde{v}, \Gamma]) + \mathrm{Tr}((\tilde{v}\Gamma)[\tilde{v}\Gamma, \Gamma])$;

(b) (b.1) $\mathrm{Tr}(-\Delta[\rho_\Gamma * \tilde{v}, \Gamma]) + \mathrm{Tr}((\rho_\Gamma * \tilde{v})[-\Delta, \Gamma])$,

(b.2) $-\mathrm{Tr}(-\Delta[\tilde{v}\Gamma, \Gamma]) - \mathrm{Tr}((\tilde{v}\Gamma)[-\Delta, \Gamma])$,

(b.3) $-\mathrm{Tr}((\rho_\Gamma * \tilde{v})[\tilde{v}\Gamma, \Gamma]) - \mathrm{Tr}((\tilde{v}\Gamma)[\rho_\Gamma * \tilde{v}, \Gamma])$;

(c) (c.1) $\mathrm{Tr}(-\Delta[\Lambda,\Lambda^*]_{\tilde{v}}) + \frac{1}{2}\mathrm{Tr}([-\Delta,\Lambda]_+(\tilde{v}\Lambda^*)) - \frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)[-\Delta,\Lambda]_+^*)$,

(c.2) $\mathrm{Tr}((\rho_\Gamma * v)[\Lambda,\Lambda^*]_{\tilde{v}})$,

(c.3) $\frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)(\tilde{v}\Lambda^*)) - \frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)(\tilde{v}\Lambda^*))$,

(c.4) $\frac{1}{2}\mathrm{Tr}([\rho_\Gamma * \tilde{v},\Lambda]_+(\tilde{v}\Lambda^*)) - \frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)[\rho_\Gamma * \tilde{v},\Lambda]_+^*)$,

(c.5) $-\mathrm{Tr}((\tilde{v}\Gamma)[\Lambda,\Lambda^*]_{\tilde{v}}) - \frac{1}{2}\mathrm{Tr}([\Gamma,\Lambda]_{\tilde{v},+}(\tilde{v}\Lambda^*)) - \frac{1}{2}\mathrm{Tr}([\Lambda,\bar{\Gamma}]_{\tilde{v},+}(\tilde{v}\Lambda^*))$

$+\frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)[\Gamma,\Lambda]_{\tilde{v},+}^*) + \frac{1}{2}\mathrm{Tr}((\tilde{v}\Lambda)[\Lambda,\bar{\Gamma}]_{\tilde{v},+})$.

In the calculation, we used the observation that for two operators $k_1$ and $k_2$,

$$\mathrm{Tr}((\rho_{k_1} * \tilde{v})k_2) = \mathrm{Tr}(k_1(\rho_{k_2} * \tilde{v})) \text{ and } \mathrm{Tr}((\tilde{v}k_1)k_2) = \mathrm{Tr}(k_1(\tilde{v}k_2)).$$

Even though $-\Delta$ is not a bounded operator, by a limiting argument, as long as $\Gamma \in \mathcal{L}^{4,1}$, the cyclicity of trace holds for every term in (a) and we are able to move $-\Delta$ around. Then every term in (a) vanishes. Each pair in (b) is zero, because of the cyclicity of trace and the formal identity

$$\mathrm{Tr}(A[B,C]) + \mathrm{Tr}(B[A,C]) = 0.$$

To show that every subgroup in (c) vanishes, we simply expand the expression and do cancellation. After cancelling duplicate terms and expressing results in integral forms, we are able to see that (c.2) (c.3) and (c.4) are zero. As for (c.1), we need to further use integration by parts. For (c.5), we need to further use the conditions that $\Gamma = \Gamma^*$ and $\Lambda^* = -\Lambda$.

Therefore the time derivative of the energy functional $E(\omega(t);\tilde{v})$ vanishes identically and we obtain the conservation law of energy. $\square$

Finally, we are able to prove the main theorem about the smooth potential case.

**Theorem 4.18.** *Suppose $\tilde{v} \in C_c^\infty(\mathbb{R}^3)$ and $\tilde{v}(x) = \tilde{v}(-x)$ for $x \in \mathbb{R}^3$, and the initial data $(\Gamma_0, \Lambda_0) \in \mathcal{S}_{ad}$ (the associated state is $\omega_0$) satisfies*

$$\Gamma_0 \in \mathcal{L}^{1,1} \quad \textit{and} \quad \Lambda_0 \in H^1,$$

*there is a global solution $(\Gamma(t), \Lambda(t))$ (the associated state is $\omega(t)$) to the Bogoliubov-de Gennes equations (4.19) and (4.20) such that*

*(i) $\Gamma(t) \in C(\mathbb{R}, \mathcal{L}^{1,1})$ and $\Lambda(t) \in C(\mathbb{R}, H^1)$;*

*(ii) $(\Gamma(t), \Lambda(t)) \in \mathcal{S}_{ad}$ for $t \in \mathbb{R}$;*

*(iii) $Tr(\Gamma(t)) = Tr(\Gamma_0)$ for $t \in \mathbb{R}$ (conservation of trace);*

*(iv) $\mathcal{E}_{BG}(\omega(t);\tilde{v}) = \mathcal{E}_{BG}(\omega_0;\tilde{v})$ for $t \in \mathbb{R}$ (conservation of energy).*

*Proof.* According to Lemma 4.10, let $\{(\Gamma_{k0}, \Lambda_{k0})\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{S}_{ad}$ converging to $(\Gamma_0, \Lambda_0)$ in the sense

$$\Gamma_{k0} \in \mathcal{L}^{4,1},\ \Lambda_{k0} \in H^4,\ \Gamma_{k0} \xrightarrow[k\to\infty]{\mathcal{L}^{1,1}} \Gamma_0,\ \Lambda_{k0} \xrightarrow[k\to\infty]{H^1} \Lambda_0.$$

By Proposition 4.16, there is a sequence of global solutions $(\Gamma_k(t), \Lambda_k(t))$ to Equation (4.19) and (4.20) satisfying $\Gamma_k(t=0) = \Gamma_{k0}$ and $\Lambda_k(t=0) = \Gamma_{k0}$. By Proposition

4.15, 4.16 and 4.17, solutions $(\Gamma_k(t), \Lambda_k(t))$ satisfy all conditions $(i) \sim (iv)$.

Using the local existence result Proposition 4.9, the solution $(\Gamma(t), \Lambda(t))$ exists on $[0, T]$ for some $T$. Next we show

$$\Gamma_k(t) \xrightarrow[k\to\infty]{\mathcal{L}^{1,1}} \Gamma(t),\ \Lambda_k(t) \xrightarrow[k\to\infty]{H^1} \Lambda(t), \quad t \in [0, T]$$

uniformly. Applying Lemma 4.8 to the difference of the Duhamel's formulation (4.21) and (4.22) for $(\Gamma_k(t), \Lambda_k(t))$ and $(\Gamma(t), \Lambda(t))$, we have

$$\begin{aligned}
&\|\Gamma_k(t) - \Gamma(t)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(t) - \Lambda(t)\|_{H^1} \\
\lesssim &\|\Gamma_{k0} - \Gamma_0\|_{\mathcal{L}^{1,1}} + \|\Lambda_{k0} - \Lambda_0\|_{H^1} \\
&+ \|\tilde{v}\|_{W^{1,\infty}} \int_0^t ds\, \big( (\|\Gamma_k(s) - \Gamma(s)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(s) - \Lambda(s)\|_{H^1}) (\|\Gamma(s)\|_{\mathcal{L}^{1,1}} + \|\Lambda(s)\|_{H^1}) \\
&+ (\|\Gamma_k(s)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(s)\|_{H^1}) (\|\Gamma_k(s) - \Gamma(s)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(s) - \Lambda(s)\|_{H^1}) \big).
\end{aligned}$$

By $(ii)$ and $(iv)$, $\|\Gamma_k(t)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(t)\|_{H^1}$ are uniformly bounded by the energy and the trace on $t \in \mathbb{R}$. Thus for $t \in [0, T]$,

$$\begin{aligned}
&\|\Gamma_k(t) - \Gamma(t)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(t) - \Lambda(t)\|_{H^1} \\
\leq &C_1 (\|\Gamma_{k0} - \Gamma_0\|_{\mathcal{L}^{1,1}} + \|\Lambda_{k0} - \Lambda_0\|_{H^1}) \\
+ &C_2 \int_0^t ds\, (\|\Gamma_k(s) - \Gamma(s)\|_{\mathcal{L}^{1,1}} + \|\Lambda_k(s) - \Lambda(s)\|_{H^1})
\end{aligned}$$

where $C_1$ and $C_2$ are constants. After applying the the Grönwall's inequality, we obtain the uniform convergence of $(\Gamma_k(t), \Lambda_k(t))$ to $(\Gamma(t), \Lambda(t))$ on $t \in [0, T]$, in the sense

$$\Gamma_k(t) \xrightarrow[k\to\infty]{\mathcal{L}^{1,1}} \Gamma(t),\ \Lambda_k(t) \xrightarrow[k\to\infty]{H^1} \Lambda(t).$$

Since all conditions $(i) \sim (iv)$ are continuous with respect to the norm $\mathcal{L}^{1,1}$ on $\Gamma$ and the norm $H^1$ on $\Lambda$, over the interval $t \in [0,T]$, $(\Gamma(t), \Lambda(t))$ satisfy all of the conditions. In addition, the trace and the energy majorize the $\mathcal{L}^{1,1}$ norm of $\Gamma(t)$. Therefore conditions $(iii)$ and $(iv)$ imply that $\|\Gamma(t)\|_{\mathcal{L}^{1,1}}$ stays bounded in time and we extend the unique mild solution $(\Gamma(t), \Lambda(t))$ globally. $\square$

## 4.5 Global Result.

In this section, we prove the main theorem 2.16.

*Proof.* Let $\{v_j\}$ be a sequence of potentials in $C_c^\infty(\mathbb{R}^3)$ such that $v_j(x) = v_j(-x)$ and $v_j(x) \geq 0$ for $x \in \mathbb{R}^3$ and the sequence converges to $v$ with respect to the norm $\|\cdot\|_M$. Such sequence can be obtained by truncating and mollifying $v$. We evolve the initial data $(\Gamma_0, \Lambda_0)$ by the Bogoliubov-de Gennes equations with potential $v_j$, i.e.

$$i\,\partial_t\Gamma(t) = [-\Delta, \Gamma(t)] + \underbrace{[v_j * \rho_{\Gamma(t)}, \Gamma(t)] - [\Gamma(t), \Gamma(t)]_{v_j} + [\Lambda(t), \Lambda^*(t)]_{v_j}}_{F_1(t;v_j)},$$

$$i\,\partial_t\Lambda(t) = [-\Delta, \Lambda(t)]_+ + (v_j\Lambda)(t)$$
$$+ \underbrace{[v_j * \rho_{\Gamma(t)}, \Lambda(t)]_+ - [\Gamma(t), \Lambda(t)]_{v_j,+} - \left[\Lambda(t), \bar{\Gamma}(t)\right]_{v_j,+}}_{F_2(t;v_j)},$$

and denote corresponding solutions by $(\Gamma_j(t), \Lambda_j(t))$ (the associated state is $\omega_j(t)$). By Theorem 4.18, $(\Gamma_j(t), \Lambda_j(t))$ exist globally and satisfies $(i) \sim (iv)$.

Using the local well-posedness result Theorem 2.15, $(\Gamma(t), \Lambda(t))$ exists over $[0, T]$ for some $T$. Applying Lemma 4.4 and Lemma 4.7 to the difference of the Duhamel's formulations for $(\Gamma_j(t), \Lambda_j(t))$ and $(\Gamma(t), \Lambda(t))$

$$\|\omega_j(t) - \omega(t)\|_{N_T} \lesssim (T^{\frac{\epsilon}{4}} + T)\,\big(\|\omega_j(t) - \omega(t)\|_{N_T}\|\omega_j\|_{N_T}\|v_j\|_M \\ + \|\omega(t)\|_{N_T}\|\omega_j(t) - \omega(t)\|_{N_T}\|v_j\|_M + \|\omega(t)\|_{N_T}^2\|v_j - v\|_M\big)$$

Since $\|\omega_j\|_{N_T}$ and $\|v_j\|_M$ are uniformly bounded, for sufficiently small $T$, we can absorb the first two terms on the right hand side of the last inequality to the left hand side and obtain

$$\frac{1}{2}\|\omega_j(t) - \omega(t)\|_{N_T} \leq C\|v_j - v\|_M \quad \text{for small } T,$$

which implies that $\omega_j(t) \xrightarrow[j\to\infty]{N_T} \omega(t)$.

The condition of a state being quasi-free is on the level of operator norms. While the norm $N_T$ includes the Hilbert-Schmidt norms

$$\|\Gamma(t)\|_{L_t^\infty([0,T],\mathcal{L}^2)} \quad \text{and} \quad \|\Lambda(t)\|_{L_t^\infty([0,T],\mathcal{L}^2)}$$

which are stronger than the operator norms. As $\omega_j(t)$ converges to $\omega(t)$ in $N_T$, the quasi-free condition passes to the $(\Gamma(t), \Lambda(t))$.

The trace is continuous to the trace norm of $\Gamma(t)$, which is included in the norm $N_T$. Therefore

$$\mathrm{Tr}(\Gamma_0) = \lim_{k\to\infty} \mathrm{Tr}(\Gamma_j(t)) = \mathrm{Tr}(\Gamma(t)), \quad t \in [0,T].$$

At the initial time, since $\Gamma_0 \in \mathcal{L}^{1,1}$ and $\Lambda_0 \in H^1$, it is clear that

$$\lim_{j\to\infty} \mathcal{E}_{BG}(\omega_0; v_j) = \mathcal{E}_{BG}(\omega_0; v).$$

However it requires extra work to show the convergence of energy for other times.

By Lemma 4.13, if $(\Gamma(t),\Lambda(t)) \in \mathcal{S}_{ad}$, $\|\Lambda\|_{H^1}^2 + \|\Gamma\|_{H^1}^2 \lesssim \|\Gamma\|_{\mathcal{L}^{1,1}}$, but the reverse inequality $\|\Gamma\|_{\mathcal{L}^{1,1}} \lesssim \|\Lambda\|_{H^1}^2 + \|\Gamma\|_{H^1}^2$ is not necessarily true for general quasi-free states. Therefore the convergence of $\omega_k(t)$ to $\omega(t)$ in $N_T$ does not imply the convergence $\|\Gamma_j(t) - \Gamma(t)\|_{L_t^\infty([0,T],\mathcal{L}^{1,1})} \to 0$. In order to show that the property $(iv)$ holds for $\omega(t)$ and $\Gamma(t) \in \mathcal{L}^{1,1}$ for $t \in [0,T]$, we split the total energy $E(\omega; v)$ into $E_1(\omega(t); v)$ and $E_2(\omega(t))$:

$$\begin{aligned} E_1(\omega(t); v) &:= \frac{1}{2}\left\{-\mathrm{Tr}\left((v\Gamma)(t)\Gamma^*(t)\right) + \mathrm{Tr}\left((v\Lambda)(t)\Lambda^*(t)\right)\right\} \\ E_2(\omega(t); v) &:= \mathrm{Tr}(|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}) + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y). \end{aligned}$$

Employing the fixed time estimate for $\|(v\Gamma)(t)\Gamma^*(t)\|_{\mathcal{L}^1}$ and $\|(v\Lambda)(t)\Lambda^*(t)\|_{\mathcal{L}^1}$ (see the proof of Lemma 4.4),

$$\|(v\Gamma)(t)\Gamma^*(t)\|_{\mathcal{L}^1} \lesssim \|v\|_M \|\omega(t)\|_{N_T}^2 \quad \text{and} \quad \|(v\Lambda)(t)\Lambda^*(t)\|_{\mathcal{L}^1} \lesssim \|v\|_M \|\omega(t)\|_{N_T}^2,$$

we obtain the convergence for the $E_1$ part

$$\lim_{j\to 0} E_1(\omega_j(t); v_j) = E_1(\omega(t); v).$$

The proof of the convergence of the $E_2$ part and $\Gamma(t) \in \mathcal{L}^{1,1}$ depends on the observations that $\Gamma(t)$ is positive, $v_j(x) \geq 0, v(x) \geq 0$ for $x \in \mathbb{R}^3$ and $\rho_{\Gamma(t)}(x) \geq 0$. It consists of the following two steps

Step 1 $|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}$ is well defined. Since $\Gamma_j(t)$ converges to $\Gamma(t)$ in operator norm, for any $f, g \in H^1(\mathbb{R}^3)$ with $\|f\|_{L^2} = \|g\|_{L^2} = 1$

$$\lim_{j\to\infty} \left\langle |\Delta|^{1/2}\Gamma_j(t)|\Delta|^{1/2} f, g\right\rangle = \lim_{j\to\infty} \left\langle \Gamma_j(t)|\Delta|^{1/2} f, |\Delta|^{1/2} g\right\rangle = \left\langle \Gamma(t)|\Delta|^{1/2} f, |\Delta|^{1/2} g\right\rangle.$$

And we have the estimate

$$\begin{aligned}
&\left|\left\langle |\Delta|^{1/2}\Gamma_j(t)|\Delta|^{1/2} f, g\right\rangle\right| \\
\leq &\||\Delta|^{1/2}\Gamma_j(t)|\Delta|^{1/2}\|_{\mathcal{L}^1} = \mathrm{Tr}\left(|\Delta|^{1/2}\Gamma_j(t)|\Delta|^{1/2}\right) \\
\leq &\mathcal{E}_{BG}(\omega_j(t); v_j) - E_1(\omega_j; v_j) = \mathcal{E}_{BG}(\omega_0; v_j) - E_1(\omega_j(t); v_j)
\end{aligned}$$

Therefore as $j \to \infty$,

$$\left|\left\langle \Gamma(t)|\Delta|^{1/2} f, |\Delta|^{1/2} g\right\rangle\right| \leq E(\omega_0; v) - E_1(\omega(t); v) \tag{4.25}$$

Since $\|\omega(t)\|_{N_T} < \infty$, $|\Delta|^{1/2}\Gamma(t)$ is well-defined,

$$\left\langle \Gamma(t)|\Delta|^{1/2}f, |\Delta|^{1/2}g\right\rangle = \left\langle |\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}f, g\right\rangle$$

Because $g$ is arbitrary and $H^1(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, the boundedness (4.25) implies that $|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}f$ is well-defined and bounded. Furthermore $f$ is arbitrary, $|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}$ is a well-defined bounded positive operator.

Step 2 $E_2(\omega(t); v) = \mathcal{E}_{BG}(\omega_0; v) - E_1(\omega(t); v)$ and $\Gamma(t) \in \mathcal{L}^{1,1}$. Let $\{f_i\}_{i\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be an orthonormal basis of $L^2(\mathbb{R}^3)$. Since $v_j(x-y)\rho_{\Gamma_j(t)}(x)\rho_{\Gamma_j(t)}(y)$ converges to $v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y)$ pointwise (at least there is a sub-sequence of $\{\omega_j(t)\}_{j\in\mathbb{N}}$ satisfies the requirement), using the Fatou's lemma,

$$\begin{aligned}
&\limsup_{j\to\infty} E_2(\omega_j(t); v_j)\\
&= \limsup_{j\to\infty}\left(\sum_{i\in\mathbb{N}}\left\langle |\Delta|^{1/2}\Gamma_j(t)|\Delta|^{1/2}f_i, f_i\right\rangle + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v_j(x-y)\rho_{\Gamma_j(t)}(x)\rho_{\Gamma_j(t)}(y)\right)\\
&\le \sum_{i\in\mathbb{N}}\limsup_{j\to\infty}\left\langle \Gamma_j(t)|\Delta|^{1/2}f_i, |\Delta|^{1/2}f_i\right\rangle\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, \limsup_{j\to\infty} v_j(x-y)\rho_{\Gamma_j(t)}(x)\rho_{\Gamma_j(t)}(y)\\
&= \sum_{i\in\mathbb{N}}\left\langle |\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}f_i, f_i\right\rangle + \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\liminf_{j\to\infty} E_2(\omega_j(t); v_j) \ge &\sum_{i\in\mathbb{N}}\left\langle |\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}f_i, f_i\right\rangle\\
&+ \frac{1}{2}\int_{\mathbb{R}^6} dxdy\, v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y).
\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{i\in\mathbb{N}}\left\langle|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}f_i,f_i\right\rangle+\frac{1}{2}\int_{\mathbb{R}^6}dxdy\,v(x-y)\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y)\\ =&\mathcal{E}_{BG}(\omega_0;v)-E_2(\omega(t);v),\end{aligned}$$

and $|\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}\in\mathcal{L}^1$ using the property of bounded positive operators [Sim05, Theorem 2.14].

The total energy $E(\omega(t);t)$ bounds $\||\Delta|^{1/2}\Gamma(t)|\Delta|^{1/2}\|_{\mathcal{L}^1}$ since the potential energy part is non-negative. Given fixed time $t$, if $\Gamma(t)$ is of trace class and positive, it can be written as $\Gamma(t,x,y)=\sum_{i\in\mathbb{N}}\lambda_i f_i(x)\bar{f}_i(y)$ for some orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$ of $L^2(\mathbb{R}^3)$ and $\lambda_i\geq 0$, and we have the pointwise inequality

$$\begin{aligned}&\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y)\\ =&\left(\sum_{i\in\mathbb{N}}\lambda_i|f_i|^2(x)\right)\left(\sum_{i\in\mathbb{N}}\lambda_i|f_i|^2(y)\right)\geq\left(\sum_{i\in\mathbb{N}}\lambda_i|f_i|(x)|f_i|(y)\right)^2\\ \geq&|\Gamma|^2(t,x,y).\end{aligned}$$

Thus the sum

$$\int_{\mathbb{R}^6}dxdy\,v(x-y)\left(\rho_{\Gamma(t)}(x)\rho_{\Gamma(t)}(y)-|\Gamma|^2(t,x,y)\right)\geq 0,$$

and the potential energy part of the total energy is non-negative. Therefore we can bound $\|\Gamma(t)\|_{\mathcal{L}^{1,1}}$

$$\|\Gamma(t)\|_{\mathcal{L}^{1,1}}\leq\mathrm{Tr}(\Gamma(t))+\mathcal{E}_{BG}(\omega(t);v).$$

So far we have shown that $(\Gamma(t), \Lambda(t))$ satisfies $(i) \sim (iv)$ over the interval $[0, T]$. Using the uniform boundedness of $\|\Gamma(t)\|_{\mathcal{L}^{1,1}}$, we can extend the solution to an larger interval and repeat the same argument. Furthermore $\|\Gamma(t)\|_{\mathcal{L}^{1,1}}$ stays bounded in time, repeating the process, we extend the solution $(\Gamma(t), \Lambda(t))$ over $\mathbb{R}$. $\square$

## 4.6 Appendix

The proof of the Morrey's inequality for Banach spaces is based on the classical argument for the scalar case [Eva10, Chapter 5, Theorem 4].

**Proposition 4.19** (Morrey's inequality)**.** *Let $u \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ and $p > n$. For every $x \in \mathbb{R}^n$, $u(x, \cdot)$ is valued in a Banach space with norm $B$. Then*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n, B)} \lesssim_{n,p} \|u\|_{W^{1,p}(\mathbb{R}^n, B)}, \tag{4.26}$$

*where $\gamma = 1 - n/p$.*

*Proof.* 1. Control the oscillation of $u$ in a neighborhood by $Du$, i.e. for any $x, y \in \mathbb{R}^n$

$$⨍_{B_r(0)} \|u(x+y, \tilde{x}) - u(x, \tilde{x})\|_B \, dy \leq \frac{1}{\alpha(n)} \int_{B_r(0)} \frac{\|Du(x+y, \tilde{x})\|_B}{|y|^{n-1}} \, dy, \tag{4.27}$$

where $\alpha(n)$ is the area of the unit sphere in $\mathbb{R}^n$. To show (4.27), by the fundamental theorem of calculus,

$$u(x+y, \tilde{x}) - u(x, \tilde{x}) = \int_0^1 Du(x+sy, \tilde{x}) \cdot y \, ds.$$

Using the Minkowski inequality for a Banach space, we have

$$\|u(x+y,\tilde{x})-u(x,\tilde{x})\|_B \le \int_0^1 \|Du(x+sy,\tilde{x})\cdot y\|_B\,ds \le \int_0^1 \|Du(x+sy,\tilde{x})\|_B|y|\,ds.$$

Integrating the inequality over the sphere $\partial B_\rho(0)$, where $0<\rho\le r$,

$$\begin{aligned}
&\int_{\partial B_\rho(0)} \|u(x+w,\tilde{x})-u(x,\tilde{x})\|_B dw\\
\le&\int_{\partial B_\rho(0)} dw \int_0^1 \|Du(x+sw,\tilde{x})\|_B\,|w|\,ds\\
=&\int_{\partial B_1(0)} dw \int_0^1 \|Du(x+s\rho w,\tilde{x})\|_B\,\rho^n\,ds\\
=&\rho^{n-1}\int_{\partial B_1(0)} dw \int_0^1 \|Du(x+s\rho w,\tilde{x})\|_B\,(s\rho)^{n-1}\frac{1}{(s\rho)^{n-1}}\,\rho ds\\
=&\rho^{n-1}\int_{B_\rho(0)} \frac{\|Du(x+y,\tilde{x})\|_B}{|y|^{n-1}}\,dy\\
\le&\rho^{n-1}\int_{B_r(0)} \frac{\|Du(x+y,\tilde{x})\|_B}{|y|^{n-1}}\,dy,
\end{aligned}$$

then over the ball $B_r(0)$,

$$\begin{aligned}
&\int_{B_r(0)} \|u(x+y,\tilde{x})-u(x,\tilde{x})\|_B\,dy\\
\le&\int_0^r \rho^{n-1}\,d\rho \int_{B_r(0)} \frac{\|Du(x+y,\tilde{x})\|_B}{|y|^{n-1}}\,dy\\
=&\frac{\mathrm{Vol}\,(B_r(0))}{\alpha(n)}\int_{B_r(0)} \frac{\|Du(x+y,\tilde{x})\|_B}{|y|^{n-1}}\,dy.
\end{aligned}$$

2. Notice that $\|u(x,,\tilde{x})\|_B \le |\|u(y,,\tilde{x})\|_B - \|u(x,\tilde{x})\|_B| + \|u(y,\tilde{x})\|_B$ and take

the averages with respect to $y$ over the ball $B_r(x)$,

$$
\begin{aligned}
&\|u(x,\tilde{x})\|_B \\
&\leq \fint_{B_r(x)} \left| \|u(y,\tilde{x})\|_B - \|u(x,\tilde{x})\|_B \right| dy + \fint_{B_r(x)} \|u(y,\tilde{x})\|_B \, dy \\
&\leq \frac{1}{\alpha(n)} \int_{B_r(0)} \frac{\|Du(x+y,\tilde{x})\|_B}{|y|^{n-1}} \, dy + \fint_{B_r(x)} \|u(y,\tilde{x})\|_B \, dy \quad \text{(by (4.27))} \\
&\leq \frac{r^\gamma}{\alpha(n)^{1/p}} \left(\frac{p-n}{p-1}\right)^{1-1/p} \|Du\|_{L^p(B_r(x),B)} + \frac{1}{\mathrm{Vol}\,(B_r(x))^{1/p}} \|u\|_{L^p(B_r(x),B)} \\
&\quad \text{(by Hölder inequality).}
\end{aligned}
$$

Therefore $\|u\|_{L^\infty(\mathbb{R}^\infty,B)} \lesssim_{n,p} \|u\|_{W^{1,p}(\mathbb{R}^n,B)}$.

3. Control the semi-Hölder norm $[u]_{C^{0,\gamma}(\mathbb{R}^n,B)}$ by $\|Du\|_{L^p(\mathbb{R}^n,B)}$, i.e.

$$
\frac{\|u(y,\tilde{x}) - u(x,\tilde{x})\|_B}{|y-x|^\gamma} \leq \frac{a^\gamma}{\alpha(n)^{1/p} C_a} \left(\frac{p-n}{p-1}\right)^{1-1/p} \|Du\|_{L^p(B_{2r}(x),B)}\,, \tag{4.28}
$$

where $r = |y-x|$, $a$ is a fixed ratio $0 < a < 1$ and $C_a$ is a constant to be defined later. Let $U(a)$ be the intersection of two balls $B_{ar}(y)$ and $B_{ar}(x)$. To show the estimate, for any $z \in U_a$, by the triangle inequality,

$$
\|u(y) - u(x)\|_B \leq \|u(y) - u(z)\|_B + \|u(z) - u(x)\|_B.
$$

Integrating the inequality with respect to variable $z$ over $U_a$,

$$
\begin{aligned}
&\mathrm{Vol}\,(U_a)\,\|u(y,\tilde{x})-u(x,\tilde{x})\|_B\\
&\leq \int_{U_a} dz\ \|u(y,\tilde{x})-u(z,\tilde{x})\|_B + \int_{U_a} dz\ \|u(z,\tilde{x})-u(x,\tilde{x})\|_B\\
&\leq \int_{B_r(y)} dz\ \|u(y,\tilde{x})-u(z,\tilde{x})\|_B + \int_{B_r(x)} dz\ \|u(z,\tilde{x})-u(x,\tilde{x})\|_B\\
&\leq \frac{\mathrm{Vol}\,(B_{ar}(0))}{\alpha(n)}\left(\int_{B_{ar}(y)} dz\,\frac{\|Du(z,\tilde{x})\|_B}{|z-y|^{n-1}} + \int_{B_{ar}(x)} dz\,\frac{\|Du(z,\tilde{x})\|_B}{|z-x|^{n-1}}\right) \quad \text{(by (4.27))}\\
&\leq \frac{\mathrm{Vol}\,(B_{ar}(0))}{\alpha(n)}\left\|\frac{1}{|x|^{n-1}}\right\|_{L^{\frac{p}{p-1}}(B_{ar}(0,B))}\left(\|Du\|_{L^p(B_{ar}(y))} + \|Du\|_{L^p(B_{ar}(x),B)}\right)\\
&= \frac{\mathrm{Vol}\,(B_{ar}(0))\,a^{1-n/p}}{\alpha(n)^{1/p}}\left(\frac{p-n}{p-1}\right)^{1-1/p} r^{1-n/p}\|Du\|_{L^p(B_{2r}(x),B)}.
\end{aligned}
$$

The volume of the intersection $U_a = B_{ar}(x)\cap B_{ar}(y)$ has a fixed ratio with respect to $\mathrm{Vol}\,(B_{ar}(0))$ and denote it by $C_a$, i.e. $C_a = \frac{\mathrm{Vol}(U_a)}{\mathrm{Vol}(B_{ar}(0))}$. □

Next we prove Lemma 4.12.

*Proof.* Let $\Gamma(t)\in\mathcal{L}^{2,1}$ and $\Lambda(t)\in H^2$ for $t\in[-T,T]$, and $(\Gamma(t),\Lambda(t))$ be solution to the Equations (4.19) and (4.20), the associated generalized one particle matrix be $S_{\omega(t)}$. Following the same computation in [BSS18, Section 5.7.1.], $S_{\omega(t)}$ satisfies the time evolution equation

$$
i\partial_t S_{\omega(t)} = \left[\begin{pmatrix}-\Delta & 0\\ 0 & \Delta\end{pmatrix} + \begin{pmatrix}\tilde{v}*\rho_{\Gamma(t)}-(\tilde{v}\Gamma)(t) & (\tilde{v}\Lambda)(t)\\ (\tilde{v}\Lambda^*)(t) & \tilde{v}*\rho_{\Gamma(t)}-(\tilde{v}\bar{\Gamma})(t)\end{pmatrix}, S_{\omega(t)}\right], \tag{4.29}
$$

where $t\in[-T,T]$. Split the linear operator in Equation (4.29) into an unbounded time-independent and a bounded time-dependent part as

$$A := \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} \quad \text{and} \quad V_{\omega(t)} := \begin{pmatrix} \tilde{v} * \rho_{\Gamma(t)} - (\tilde{v}\Gamma)(t) & (\tilde{v}\Lambda)(t) \\ (\tilde{v}\Lambda^*)(t) & \tilde{v} * \rho_{\Gamma(t)} - (\tilde{v}\bar{\Gamma})(t) \end{pmatrix}$$

Then apply the classical Kato-Yosida result [Kat53] to the equation

$$i\partial_t U(t,s) = \left(A + V_{\omega(t)}\right) U(t,s), \quad U(s,s) = 1,$$

and show the existence of one parameter unitary subgroup $U(t,s)$. We need to verify

1. Given a fixed $t \in [-T, T]$, $-i\left(A + V_{\omega(t)}\right)$ and $i\left(A + V_{\omega(t)}\right)$ are generators of contraction semigroups on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Since $A$ is essentially self-adjoint and $V_{\omega(t)}$ is a bounded self-adjoint operator. Given fixed time $t$, using the Kato-Rellich theorem, $A + V_{\omega(t)}$ is also essentially self-adjoint.

2. The domain $\mathcal{D}\left(A + V_{\omega(t)}\right)$ is independent of $t$. It follows from $\mathcal{D}\left(A + V_{\omega(t)}\right) = \mathcal{D}(A)$.

3. The regularity assumptions $C_2, C_3, C_4$ [Kat53] on $t \mapsto \left(A + V_{\omega(t)}\right)$. According to the recent characterization [SG14], the regularity assumptions are equivalent to the condition that for every $x \in \mathcal{D}(A)$, $t \mapsto \left(A + V_{\omega(t)}\right)x$ is continuously

differentiable. The condition is straightforward to verify since

$$\Gamma(t) \in C([-T,T],\mathcal{L}^{2,1}) \cap C^1([-T,T],\mathcal{L}^1),$$

$$\Lambda(t) \in C([-T,T],H^2) \cap C^1([-T,T],L^2),$$

and $\tilde{v} \in C_c^\infty(\mathbb{R}^3)$.

Therefore $S_{\omega(t)} = U(t,0)S_{\omega_0}U(-t,0)$ and the spectrum of $S_{\omega(t)}$ is preserved. $\square$

Following [Ara71], the lifting procedure is summarized in the proof of the next Lemma,

**Lemma 4.20.** *Let $S \in \mathcal{S}_{ad}$, there is a quasi-free state $\omega$ such that its generalized one-particle matrix $S_\omega = S$.*

*Proof.* Let $S \in \mathcal{S}_{ad}$. For simplicity, let $\mathcal{H}_{\mathbb{C}}$ denote the Hilbert space $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with the complex conjugation $\mathcal{J}$. Consider the $C^*$-algebra $\mathfrak{U}_{CAR}$ generated by $a^\dagger(f) + a(g)$, where $(f,g) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, and denote its completion with respect to the $C^*$-norm [SS64, Proposition 1] by $\overline{\mathfrak{U}}_{CAR}(\mathcal{H}_{\mathbb{C}})$.

1. When $S$ is a matrix such that $S^2 = S$, by virtue of [Ara71, Lemma 4.3] and $\mathcal{F}_a$ being the irreducible representation of $\overline{\mathfrak{U}}_{CAR}(\mathcal{H}_{\mathbb{C}})$ [Coo53], there is a Fock state $\psi \in \mathcal{F}_a$ such that $\omega = |\psi\rangle\langle\psi|$ and $S_\omega = S$. Every non-trivial Fock state of $\mathcal{F}_a$ is cyclic and the collection of all correlation functions is equivalent to a positive functional $u \mapsto \langle \psi, u\psi\rangle_{\mathcal{F}_a}$, $u \in \overline{\mathfrak{U}}_{CAR}(\mathcal{H}_{\mathbb{C}})$. Using a classical result in $C^*$-algebra, $\psi$ is uniquely up to phases and $\omega$ is uniquely determined as a pure state.

2. When $S$ does not satisfies the equation $S^2 = S$, by [Ara71, Lemma 4.5, 4.6], we lift $S$ to

$$P_S = \begin{pmatrix} S & S^{1/2}(id - S)^{1/2} \\ S^{1/2}(id - S)^{1/2} & id - S \end{pmatrix}$$

over the space $\hat{\mathcal{H}}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}}$ with complex conjugation $\hat{\mathcal{J}} = \mathcal{J} \oplus (-\mathcal{J})$, where $id$ is the identity on $\mathcal{H}_{\mathbb{C}}$. Note that $S^{1/2}(id - S)^{1/2}$ commutes with $\mathcal{J}$, $P_S + \hat{\mathcal{J}} P_S \hat{\mathcal{J}} = id_{\hat{\mathcal{H}}_{\mathbb{C}}}$, $P_S^* = P_S = P_S^2$. We reduce to Case 1. Now the Fock representation of $\hat{\mathcal{H}}_{\mathbb{C}}$ is over $\hat{\mathcal{F}}_a$, which is generated by $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. Using the standard result in Clifford algebras [LM89, Proposition 1.5], $\hat{\mathcal{F}}_a$ can be regarded as

$$\hat{\mathcal{F}}_a = \mathcal{F}_{a1} \hat{\otimes} \mathcal{F}_{a2},$$

where $\hat{\otimes}$ is a $\mathbb{Z}_2$-graded tensor product. The splitting is orthogonal. $\mathcal{F}_{a1}$ and $\mathcal{F}_{a1}$ are identical copies of $\mathcal{F}_a$. We use subscripts 1 and 2 just to specify which copy we refer to. Since the complex conjugation of $\hat{\mathcal{H}}_{\mathbb{C}}$ is $\hat{\mathcal{J}}$ and note the compatibility condition [Ara71, Section 2 Notations], we realize

$$(f_1, g_1) \oplus (f_2, g_2) \mapsto a_1^\dagger(f_1) + a_1(g_1) + ia_2(f_2) + ia_2^\dagger(g_2)$$

where $(f_1, g_1) \oplus (f_2, g_2) \in \hat{\mathcal{H}}_{\mathbb{C}}$. Let $\pi_1$ denote the projection of $\hat{\mathcal{F}}_a$ onto $\mathcal{F}_{a1}$. Apply the result for Case 1, there is a unique Fock state $\hat{\psi} \in \hat{\mathcal{F}}_a$ up to a phase, $\|\hat{\psi}\|_{\hat{F}_a} = 1$ whose generalized one-particle matrix is $P_S$. Project $\hat{\psi}_j$ to $\mathcal{F}_{a1}$, which acts on any $f \in \mathcal{F}_{a1}$ in the way

$$\left\langle \hat{\psi}_j, f \hat{\otimes} 1 \right\rangle_{\hat{\mathcal{F}}_a} \pi_1 \left( \hat{\psi}_j \right).$$

This action correspond to a quasi-free state $\omega$ on $\mathcal{F}_a$. More precisely, decompose $\hat{\psi}$

$$\hat{\psi} = \sum_{j=1}^{\infty} \lambda_j \psi_j \hat{\otimes} \phi_j,$$

where $\psi_j \hat{\otimes} \phi_j$ are in the form $e_j \hat{\otimes} f_k$, $e_j$ and $f_k$ are orthonormal in $\mathcal{F}_a$. Then $\omega$ assumes the form

$$\omega = \sum_{j=1}^{\infty} |\lambda_j|^2 |\psi_j\rangle \langle \psi_j|$$

where $\sum_{j=1}^{\infty} |\lambda_j|^2 = 1$.

□

# Chapter 5: Conclusion and Discussion

The Hartree equation as a reduced version of the Hartree-Fock equation demonstrates distinct properties: it admits stationary solutions, which serve as formal Fermi sea of the system. We studied the Hartree equation for the perturbation of the stationary solution when there is a constant background magnetic field in the many-body system. To the best of my knowledge, in the presence of a constant magnetic field, we are the first one to consider the Hartree equation for the perturbation of the stationary solution. The formulation is a mathematical model for a many-body system with infinitely many electrons, while the main part is at low energy state and the other part is highly excited.

The problem was originally addressed in dimension three

$$i\,\partial_t \Gamma(t) = \left[\mathfrak{h} + \rho_{\Gamma(t)} * V, \Gamma(t)\right].$$

As a first step to attack the problem, we considered a two-dimensional version of the Hartree equation, which captures the discrete feature of the original problem. Since the stationary solution is not of trace class and the forcing term is not small a priori, we introduced the Fourier-Wigner transform and derived an estimate on the asymptotic behavior of associated Laguerre polynomials to obtain a collapsing estimate for the density term. Using the estimate, we proved that the two-dimensional version is locally well-posed for the perturbation of the stationary solution.

The next goal is to consider the original three-dimensional Hartree equation. The one-particle Hamiltonian $\mathfrak{h}$ of the equation has a mixed feature: the Hamiltonian $\mathfrak{h}$ has discrete spectrum when it is restricted to the first two dimension and has continuous spectrum when it is restricted to the third dimension. Since the discrete and continuous part of the one-particle Hamiltonian $\mathfrak{h}$ can not be analyzed independently when we consider the density term of the pertubation, it requires to develop further machinery to obtain the corresponding collapsing estimate and the well-posedness theory for the equation.

Another interesting direction of the problem is to study the many-body system under local magnetic field or perturbation to constant magnetic field. The local magnetic field case could be considered as perturbation of the Laplace case and the other one could be considered as perturbation of the constant magnetic field case. They are more general than the original settings and might have interesting physics applications.

Recently, the Bogoliubov-de Gennes equations were derived as an application of the Dirac-Frenkel approximation principle to pure quasi-free states by Benedikter-Sok-Solovej [BSS18]. The evolution of two-particle correlation functions for mixed quasi-free states is also described by the Bogoliubov-de Gennes equations. They provide an approximation scheme to the dynamics of the Fermionic many-body system when the initial state of the system is quasi-free. [BSS18]The existing global well-posedness theory of the Bogoliubov-de Gennes equations is for the Coulomb potential. The result is based on the semi-group theory. We employed the dispersive PDE techniques and the observation that the pairing function is anti-symmetric to extend the global well-posedness theory for more singular potentials such as $\frac{1}{|x|^{2-\epsilon}}$, for any $0 \leq \epsilon < 2$. The future work is to compare the dynamics governed by the Bogoliubov-de Gennes equations with the dynamics governed by the many-body Schrödinger equation in the mean field regime. Not all mixed states will be taken into consideration. Inspired by [GM13, GM17], we may expect to start with pure quasi-free states generated by pair excitations or mixed quasi-free states which are derived by projecting pure quasi-free states generated by pair excitations in the double-Fock representation.

# Chapter 6: Appendix

There has been a long history of studying the $C^*$-algebra CAR. Nowadays, it has been a standard content in quantum physics and mathematics. We intend to focus on the standard Fock representation of CAR and the Pin group representation instead of reviewing the vast literature. The spinor representations of infinite orthogonal groups was constructed by Shale-Stinespring [SS65]. In the appendix, we presented the result and obtained some analysis approximation results.

The Pin group representation is closely related to pure quasi-free states, whose expected number of particles are finite. Even though unitary implementable Bogoliubov transforms corresponds to all such pure quasi-free states, the Pin group representation forms an important subspace of the space of unitary implementable Bogoliubov transforms. And they provide ideas to approximate unitary implementable Bogoliubov transforms. We refer interested readers to [LM89] for the background of Clifford algebras and [SS64, BV68, Seg47, Ara71, Ara69, PSr70] for the discussion of CAR and states.

## 6.1 Fock Space

In quantum physics, the state space of a single particle is a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. When we study many-body problems, the state space is then the tensor products of Hilbert spaces. Following the construction in [Coo53], the Fock space $\mathcal{F}$ over $\mathcal{H}$ is the complete tensor algebra

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \tag{6.1}$$

where $\mathcal{H}^{\otimes n}$ denotes the tensor product of $n$ copies of $\mathcal{H}$ and it is the state space of $n$ quantum particles. The vacuum of the Fock space $\mathcal{F}$ is a state

$$|0\rangle := (1, 0, 0, \ldots), \tag{6.2}$$

where 1 is the constant in $\mathbb{C}$. The Fock space $\mathcal{F}$ derives the inner product structures from $\mathcal{H}^{\otimes n}$. Let $\varphi = (\varphi^0, \varphi^1, \ldots)$ and $\phi = (\phi^0, \phi^1, \ldots)$ be Fock states, i.e. $\varphi, \phi \in \mathcal{F}$, the inner product on $\mathcal{F}$ is defined as

$$\langle\varphi, \phi\rangle_{\mathcal{F}} := \sum_{j=0}^{\infty} \langle\varphi^j, \phi^j\rangle_{\mathcal{H}^{\otimes n}}. \tag{6.3}$$

We introduce creation and annihilation operators to the Fock space $\mathcal{F}$, which connect subspaces of $\mathcal{F}$ with different grades. The operators are defined on decomposable tensors in the following way and extend linearly to $\mathcal{F}$: the annihilation operator

$a : \mathcal{H} \times \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$, i.e. lowering the grade

$$a(f) : f_1 \otimes \cdots \otimes f_{n+1} \mapsto \langle f, f_1 \rangle_{\mathcal{H}} f_2 \otimes \cdots \otimes f_{n+1}, \quad f, f_j \in \mathcal{H},$$

and the creation operator $a^\dagger : \mathcal{H} \times \mathcal{H}^{\otimes(n-1)} \to \mathcal{H}^{\otimes n}$, i.e. raising the grade

$$a^\dagger(f) : f_1 \otimes \cdots \otimes f_{n-1} \mapsto f \otimes f_1 \otimes \cdots \otimes f_{n-1}, \quad f, f_j \in \mathcal{H}.$$

There are two types of particles in quantum physics: Fermions and Bosons. Mathematically, they correspond to two types of symmetry: anti-symmetry and symmetry respectively, and are modeled by two quotient spaces of $\mathcal{F}$. Fermions are modeled by anti-symmetric tensors

$$\mathcal{F}_a := \mathcal{F}/\mathcal{I}\left(\{f \otimes g + g \otimes f : f, g \in \mathcal{H}\}\right), \tag{6.4}$$

where $\mathcal{I}\left(\{f \otimes g + g \otimes f : f, g \in \mathcal{H}\}\right)$ denotes the ideal of $\mathcal{F}$ generated by $\{f \otimes g + g \otimes f : f, g \in \mathcal{H}\}$. Bosons are modeled by symmetric tensors

$$\mathcal{F}_s := \mathcal{F}/\mathcal{I}\left(\{f \otimes g - g \otimes f : f, g \in \mathcal{H}\}\right), \tag{6.5}$$

where $\mathcal{I}\left(\{f \otimes g - g \otimes f : f, g \in \mathcal{H}\}\right)$ denotes the ideal of $\mathcal{F}$ generated by $\{f \otimes g - g \otimes f : f, g \in \mathcal{H}\}$. Since $\mathcal{F}_s$ and $\mathcal{F}_a$ are quotient algebras of $\mathcal{F}$. They inherit multiplication structure from $\mathcal{F}$ directly. The multiplication structures are called symmetric tensor product and wedge product respectively. We can also regard $\mathcal{F}_a$ as a subspace of $\mathcal{F}$

through the embedding

$$\begin{aligned}\iota_a : f_1 \wedge \cdots \wedge f_n \mapsto & \sqrt{n!}\,\mathrm{Asym}(f_1 \otimes \cdots \otimes f_n) \\ & := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},\end{aligned}$$

and $\mathcal{F}_s$ as a subspace of $\mathcal{F}$ through the embedding

$$\begin{aligned}\iota_s : f_1 \otimes_s \cdots \otimes_s f_n \mapsto & \sqrt{n!}\,\mathrm{Sym}(f_1 \otimes \cdots \otimes f_n) \\ & := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},\end{aligned}$$

where $S_n$ is the symmetric group of $d$ elements and $\mathrm{sgn}(\sigma)$ denotes the sign of $\sigma$. Let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}$. Through the embeddings, the pull back inner product on $\mathcal{F}_a$ is characterized by the orthonormal basis

$$\{e_{k_1} \wedge \cdots \wedge e_{k_n}\}_{k_1 < \cdots < k_n},$$

and the pull back inner product on $\mathcal{F}_s$ is characterized by the orthonormal basis

$$\left\{ \frac{1}{\sqrt{n_1!}} \cdots \frac{1}{\sqrt{n_j!}} e_{k_1}^{\otimes_s n_1} \otimes_s \cdots \otimes_s e_{k_j}^{\otimes_s n_j} \right\}_{k_1 < \cdots < k_j}.$$

The above identification of $\mathcal{F}_a$ and $\mathcal{F}_s$ as subspaces of $\mathcal{F}$ are not sections from $\mathcal{F}_a$ and $\mathcal{F}_a$ to $\mathcal{F}$, namely the composition

$$\mathcal{F}_s \xrightarrow{\iota_s} \mathcal{F} \to \mathcal{F}_s$$

is not an identity on $\mathcal{F}_s$. The induced creation and annihilation operators on $\mathcal{F}_a$

$$a^\dagger(f)(f_1 \wedge \cdots \wedge f_{n-1}) = f \wedge f_1 \wedge \cdots \wedge f_{n-1},$$
$$a(f)(f_1 \wedge \cdots \wedge f_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j+1} \langle f, f_j \rangle_{\mathcal{H}} f_1 \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_{n+1},$$

where $f, f_j \in \mathcal{H}$, on $\mathcal{F}_s$

$$a^\dagger(f)(f_1 \otimes_s \cdots \otimes_s f_{n-1}) = f \otimes_s f_1 \otimes_s \cdots \otimes_s f_{n-1}$$
$$a(f)(f_1 \otimes_s \cdots \otimes_s f_{n+1}) = \sum_{j=1}^{n+1} \langle f, f_j \rangle_{\mathcal{H}} f_1 \otimes_s \cdots \otimes_s \hat{f}_j \otimes_s \cdots \otimes_s f_{n+1}.$$

where $\hat{f}_j$ means the $f_j$ is omitted. In both cases, $a^\dagger(f)$ is adjoint to $a(f)$, i.e. $(a^\dagger(f))^* = a(f)$.

In the Fermionic case, $a^\dagger(f)$ is a bounded operator, where $f \in \mathcal{H}$. Since let $\phi^n \in \bigwedge^n \mathcal{H}$,

$$\left\| a^\dagger(f)(\phi^n) \right\|_{\mathcal{F}_a} \le \|f\|_{\mathcal{H}} \|\phi^n\|_{\mathcal{F}_a},$$

$\|a^\dagger(f)\|_{op} \le \|f\|_{\mathcal{H}}$. However in the Bosonic case, $a^\dagger(f)$ is not a bounded operator. Because if $\phi^n \in \mathcal{H}^{\otimes_s n}$,

$$\left\| a^\dagger(f)(\phi^n) \right\|_{\mathcal{F}_s} \le \sqrt{n+1}\, \|f\|_{\mathcal{H}} \|\phi^n\|_{\mathcal{F}_s}.$$

In some sense, that $a^\dagger(f)$ is not bounded for $\mathcal{F}_s$ is due to the condensation of particles.

In $\mathcal{F}_a$, the canonical anticommutation relations (CAR) are

$$[a(f), a(g)]_+ = 0, \ \left[a^\dagger(f), a^\dagger(g)\right]_+ = 0, \ \left[a(f), a^\dagger(g)\right] = \langle f, g\rangle_{\mathcal{H}}. \tag{6.6}$$

In $\mathcal{F}_s$, the canonical commutation relations (CCR) are

$$[a(f), a(g)] = 0, \ [a^\dagger(f), a^\dagger(g)] = 0, \ [a(f), a^\dagger(g)] = \langle f, g\rangle_{\mathcal{H}}. \tag{6.7}$$

Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator (bounded or unbounded), the second quantization $\hat{T}$ is an extension of $T$ over $\mathcal{F}$ such that it acts a slice of tensor

$$\hat{T}\left(f_1 \otimes \cdots \otimes f_n\right) = \sum_{j=1}^{n} f_1 \otimes \cdots \otimes T\left(f_j\right) \otimes \cdots \otimes f_n.$$

The action $\hat{T}$ over $F_a$ or $F_s$ is defined by replacing the tensor product of the last expression with the corresponding multiplication structure.

**Lemma 6.1.** *Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator (bounded or unbounded) and $\hat{T}$ be the second quantization of $T$, in $\mathcal{F}_s$ or $\mathcal{F}_a$*

$$\left[a^\dagger(f), \hat{T}\right] = -a^\dagger\left(Tf\right), \quad \left[a(f), \hat{T}\right] = a\left(T^* f\right). \tag{6.8}$$

*Proof.* It suffices to show the commutation relations for a decomposable Fock state of degree $n$. Consider a Fock state $\wedge_{j=1}^{n} f_j \in \mathcal{F}_a$ and $f \in \mathcal{H}$, we have

$$
\begin{aligned}
&\left[a^\dagger(f), \hat{T}\right] \bigwedge_{j=1}^{n} f_j \\
&= \sum_{j=1}^{n} f \wedge f_1 \wedge \cdots \wedge T(f_j) \wedge \cdots \wedge f_n - \hat{T}\left(f \wedge \bigwedge_{j=1}^{n} f_j\right) \\
&= -T(f) \wedge \bigwedge_{j=1}^{n} f_j,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[a(f), \hat{T}\right] \bigwedge_{j=1}^{n} f_j \\
&= \sum_{j=1}^{n} a(f)\, f_1 \wedge \cdots \wedge T(f_j) \wedge \cdots \wedge f_n - \hat{T} \sum_{j=1}^{n} (-1)^{j+1} \langle f, f_j \rangle \bigwedge_{k \neq j} f_k \\
&= \sum_{j=1}^{n} (-1)^{j+1} \langle f, T(f_j) \rangle \bigwedge_{k \neq j} f_k \\
&= \sum_{j=1}^{n} (-1)^{j+1} \langle T^*(f), f_j \rangle \bigwedge_{k \neq j} f_k.
\end{aligned}
$$

Consider a Fock state $f_1 \otimes_s f_2 \otimes_s \cdots \otimes_s f_n \in \mathcal{F}_s$ and $f \in \mathcal{H}$, we have

$$
\begin{aligned}
&\left[a^\dagger(f), \hat{T}\right] f_1 \otimes_s \cdots \otimes_s f_n \\
&= \sum_{j=1}^{n} f \otimes_s f_1 \otimes_s \cdots \otimes_s T\left(f_j\right) \otimes_s \cdots \otimes_s f_n - \hat{T} f_1 \otimes_s f_1 \otimes_s \cdots \otimes_s f_n \\
&= -T(f) \otimes_s f_1 \otimes_s \cdots \otimes_s f_n,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[a(f),\hat{T}\right] f_1 \otimes_s \cdots \otimes_s f_n \\
=&a(f)\sum_{j=1}^{n} f_1 \otimes_s \cdots \otimes_s T(f_j) \otimes_s \cdots \otimes_s f_n - \hat{T}\sum_{j=1}^{n}\langle f, f_j\rangle_{\mathcal{H}} f_1 \otimes_s \cdots \otimes_s \hat{f}_j \otimes_s \cdots \otimes_s f_n \\
=&\sum_{j=1}^{n}\langle f, T(f_j)\rangle f_1 \otimes_s \cdots \otimes_s \hat{f}_j \otimes_s \cdots \otimes_s f_n.
\end{aligned}
$$

□

A mixed state $\omega$ of $\mathcal{F}_a$ (or $\mathcal{F}_s$) is a semi-positive self-adjoint trace class operator such that $\mathrm{Tr}_{\mathcal{F}_a}(\omega) = 1$. Correlation functions of $\omega$ are defined as

$$(f_1, \dots, f_n) \mapsto \mathrm{Tr}_{\mathcal{F}_a}\left(a^{\#}(f_1)\cdots a^{\#}(f_n)\omega\right), \tag{6.9}$$

where $f_j \in \mathcal{H}$ and $a^{\#}$ denotes an operator without specifying whether it is a creation or annihilation operator. A mixed state $\omega$ of $\mathcal{F}_a$ is quasi-free if it satisfies the Wick's theorem, i.e. any of its correlation functions can be determined by the two-particle correlation functions in the way

$$\mathrm{Tr}_{\mathcal{F}_a}\left(a^{\#}(f_1)a^{\#}(f_2)\cdots a^{\#}(f_{2n+1})\omega\right) = 0 \tag{6.10}$$

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{F}_a}\left(a^{\#}(f_1)\cdots a^{\#}(f_n)\omega\right) = \sum_{\sigma\in S_{ad}} &\mathrm{sgn}(\sigma)\mathrm{Tr}_{\mathcal{F}_a}\left(a^{\#}(f_{\sigma(1)})a^{\#}(f_{\sigma(2)})\omega\right) \\
&\cdots\mathrm{Tr}_{\mathcal{F}_a}\left(a^{\#}(f_{\sigma(2n-1)})a^{\#}(f_{\sigma(2n)})\omega\right)
\end{aligned} \tag{6.11}
$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of permutation $\sigma$ and $S_{ad}$ is a subset of the symmetric group $S_{2n}$ such that

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1),\ \sigma(2k-1) < \sigma(2k).$$

## 6.2 Spin Representation

### 6.2.1 Finite Dimensional Case

In this section, we study the Clifford action $\mu$, the skew adjoint representation $\widetilde{Ad}$ and the relations to quasi-free states

$$\begin{tikzcd} Cl(V_{\mathbb{C}}, q_{\mathbb{C}}) \arrow[r, "\widetilde{Ad}"] \arrow[d, "\mu"] & GL\left(Cl(V_{\mathbb{C}}, q_{\mathbb{C}})\right) \\ \bigwedge^{*}\left(V, \langle\cdot,\cdot\rangle_V\right) & \end{tikzcd}$$

For a thorough exposition of finite-dimensional Clifford algebras, we refer to [LM89].

Let us explain all notations in the diagram. $V$ is a 2d-dimensional real vector space endowed with a non-degenerate positive quadratic form $q$. $V$ is also endowed with a compatible complex structure $J$ such that

$$q\left(Jv_1, Jv_2\right) = q(v_1, v_2),$$

where $v_1, v_2 \in V$. Therefore $V$ can regarded as a complex space with inner product

$$\langle v_1, v_2\rangle_V := q(v_1, v_2) + iq(Jv_1, v_2).$$

When the pair $(V, q)$ is used, we consider $V$ as a real vector space. When the pair

$(V, \langle\cdot,\cdot\rangle_V)$ is used, we consider $V$ as a complex vector space. Otherwise we specify which structure of $V$ is used. $V_{\mathbb{C}} := \mathbb{C}\otimes_{\mathbb{R}} V$ and $q_{\mathbb{C}} := id \otimes q$. $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$ is the Clifford algebra

$$Cl(V_{\mathbb{C}}, q_{\mathbb{C}}) := \bigoplus_{j=0}^{\infty} V_{\mathbb{C}}^{\otimes j} / \{v \otimes v - q_{\mathbb{C}}(v) : v \in V_{\mathbb{C}}\} \tag{6.12}$$

where $\{v \otimes v - q_{\mathbb{C}}(v) : v \in V_{\mathbb{C}}\}$ denotes the ideal of $\bigoplus_{j=0}^{\infty} V_{\mathbb{C}}^{\otimes j}$ generated by $v \otimes v - q_{\mathbb{C}}(v)$. $V_{\mathbb{C}}$ is endowed with inner product

$$\langle u, v\rangle_{V_{\mathbb{C}}} := 2q_{\mathbb{C}}(\bar{u}, v)$$

where $u, v \in V_{\mathbb{C}}$. $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$ is a $2^{2d}$-dimensional complex vector space and it is isomorphic to the matrix algebra algebra $Mat(2^d, \mathbb{C})$. Since the matrix algebra is simple, $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$ has a unique finite-dimensional irreducible representation. The representation is given by $\mu$. $\bigwedge^* (V, \langle\cdot,\cdot\rangle_V)$ is the Fermionic Fock space $\mathcal{F}_a$ defined in Section 6.1, when the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is $(V, \langle\cdot,\cdot\rangle_V)$. Next we define $\mu$ using a special basis of $(V, q)$. Let $\{\partial_{x_j}, \partial_{y_j}\}_{j=1}^d$ be orthonormal basis of $(V, q)$ such that

$$J\partial_{x_j} = \partial_{y_j}, \quad q = \sum_{j=1}^{d} (dx_j \otimes dx_j + dy_j \otimes dy_j)$$

Then there is a natural basis for $V_{\mathbb{C}}$,

$$\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}), \quad \partial_{\bar{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j}), \tag{6.13}$$

where $j = 1, \ldots, d$ and $q_{\mathbb{C}}$ is

$$q_{\mathbb{C}} = \frac{1}{2} \sum_{j=1}^{d} \left( dz_j \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_j \right).$$

$\{\partial_{z_j}\}_{j=1}^{d}$ is also the orthonormal basis of $(V, \langle \cdot, \cdot \rangle_V)$. Finally, the representation $\mu$ is defined as wedge products and contraction

$$\mu\left(\partial_{z_j}\right) u \coloneqq \partial_{z_j} \wedge u, \;\; u \in \wedge^* (V, \langle \cdot, \cdot \rangle_V), \tag{6.14}$$

$$\mu\left(\partial_{\bar{z}_k}\right) \left( \bigwedge_{j=1}^{l} u_j \right) \coloneqq \sum_{j=1}^{l} (-1)^{j+1} \langle \partial_{z_k}, u_j \rangle_V \bigwedge_{i \neq j} u_i, \;\; u_j \in (V, \langle \cdot, \cdot \rangle_V). \tag{6.15}$$

Since the definition preserves the quadratic form $q_{\mathbb{C}}$ in the sense

$$\mu(v_1) \mu(v_2) + \mu(v_2) \mu(v_1) = 2 q_{\mathbb{C}}(v_1, v_2)$$

where $v_1, v_2 \in V_{\mathbb{C}}$, by the universal property of Clifford algebras, $\mu$ defines a Clifford representation of $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$. The transpose $(\cdot)^t$ on $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$ is a map

$$()^t : v_1 v_2 \ldots v_{j-1} v_j \mapsto v_j v_{j-1} \ldots v_2 v_1, \quad v_1, \ldots, v_j \in V_{\mathbb{C}}.$$

Then $V_{\mathbb{C}}$ is endowed with an $*$ involution: $u^* = \bar{u}^t$. In addition, the $*$-structure of operators corresponds to the $*$-structure on $V_{\mathbb{C}}$

$$(\mu(v))^* = \mu(\bar{v}^t), \quad v \in Cl(V_{\mathbb{C}}, q_{\mathbb{C}}). \tag{6.16}$$

Using the identities $\partial_{x_j} = \partial_{z_j} + \partial_{\bar{z}_j}$ and $\partial_{y_j} = i\partial_{z_j} - i\partial_{\bar{z}_j}$, the restriction $\mu|_{Cl(V,q)}$ is a real representation for $Cl(V,q)$. Since $V$ is of finite dimension, let us define creation and annihilation operators as

$$a_j^\dagger := \mu\left(\partial_j\right), \quad a_j := \mu\left(\partial_{\bar{z}_j}\right). \tag{6.17}$$

In order to define the skew adjoint representation $\widetilde{Ad}$, consider the involution $\alpha : v \mapsto -v$, for $v \in (V,q)$. Extending $\alpha$ linearly to $Cl(V,q)$, we obtain an involution on $Cl(V,q)$, i.e. $\alpha^2 = id_{Cl(V,q)}$ and a decomposition of $Cl(V,q)$

$$Cl(V,q) = Cl^0(V,q) \bigoplus Cl^1(V,q),$$

where $Cl^j(V,q) = \{u \in Cl(V,q) | \alpha(u) = (-1)^j u\}$. In the end, the skewed adjoint representation $\widetilde{Ad}$ is

$$\widetilde{Ad}_u(w) := \alpha(u) w u^{-1}, \quad u \in Cl^\times(V,q), \quad w \in (V,q) \tag{6.18}$$

where $Cl^\times(V,q)$ denotes the Clifford group of $Cl(V,q)$, i.e. the collection of invertible elements of $Cl(V,q)$. $\widetilde{Ad}$ coincides with the usual adjoint representation on $Cl^\times(V,q) \cap Cl^0(V,q)$.

Pure quasi-free states are closely related to the Pin subgroup of $Cl^\times(V,q)$. The Pin group $Pin(V,q) \subset Cl(V,q)$ is generated by elements

$$\{v \in V | q(v) = 1\} \tag{6.19}$$

and the Spin group $Spin(V,q)$ is a subgroup of $Pin(V,q)$

$$Spin(V,q) := Pin(V,q) \cap Cl^0(V,q).$$

**Lemma 6.2.** *The restriction of the Clifford action $\mu$ on $Pin(V,q)$ is a unitary representation, i.e.*

$$\langle \mu(v)u, \mu(v)u \rangle_V = \langle u,u \rangle_V,$$

*for any $v \in (V,q)$ and $u \in (V, \langle \cdot,\cdot \rangle_V)$.*

*Proof.* For any $v \in V$ such that $q(v) = 1$ and $u \in (V, \langle \cdot,\cdot \rangle_V)$,

$$\langle \mu(v)u, \mu(v)u \rangle_V = \langle \mu^*(v)\mu(v)u, u \rangle_V = \langle \mu(\bar{v}v)u, u \rangle_V = \langle \mu(v^2)u, u \rangle_V = \langle u,u \rangle_V.$$

Since $Pin(V,q) \subset Cl(V,q)$ is generated by $\{v \in V | q(v) = 1\}$, the restriction of $\mu$ on $Pin(V,q)$ is unitary. □

Consider the restriction $\widetilde{Ad}$ on $Pin(V,q)$ ( or $(Spin(V,q))$,

**Theorem 6.3.** *[LM89, Theorem 2.9.] There are short exact sequences*

$$0 \to \{1,-1\} \to Pin(V,q) \xrightarrow{\widetilde{Ad}} O(V,q) \to 1$$

$$0 \to \{1,-1\} \to Spin(V,q) \xrightarrow{\widetilde{Ad}} SO(V,q) \to 1.$$

*where 1 is the identity map on $(V,q)$. Furthermore, $\widetilde{Ad}$ is the covering map for $SO(V,q)$ $(O(V,q))$.*

Since $Spin(V,q)$ is the double cover of $SO(V,q)$, Lie algebras $\mathfrak{spin}(V,q)$ and $\mathfrak{so}(V,q)$ are isomorphic. A coordinate-independent description of $\mathfrak{spin}(V,q)$ and the infinitesimal representation $d\widetilde{Ad}$ is

**Lemma 6.4.** *The Lie algebra $\mathfrak{spin}(V,q)$ of $Spin(V,q)$ is generated by*

$$\{v_1v_2 - q(v_1,v_2) | v_1, v_2 \in (V,q)\}.$$

*Proof.* Note that $\{v_1v_2 - q(v_1,v_2)| v_1, v_2 \in (V,q)\}$ is in the Lie algebra $\mathfrak{spin}(V,q)$. Because for any $v_1, v_2 \in (V,q)$,

$$\begin{aligned}
&\exp\left(v_1v_2 - q(v_1,v_2)\right)\\
=&1 - \frac{c^2}{2!} + \frac{c^4}{4!} + \cdots + \left(v_1v_2 - q(v_1,v_2)\right)\left(1 - \frac{c^2}{3!} + \frac{c^4}{5!} + \cdots\right)\\
=&\cos c + \frac{\sin c}{c}\left(v_1v_2 - q(v_1,v_2)\right),
\end{aligned}$$

where $c^2 = q(v_1)q(v_2) - q^2(v_1,v_2)$, and

$$\exp\left(v_1v_2 - q(v_1,v_2)\right)\exp\left(v_1v_2 - q(v_1,v_2)\right)^t = 1.$$

Since $Spin(V,q)$ is the double cover of $SO(V,q)$, $Spin(V,q)$ is a Lie group of dimension $\binom{2d}{2}$. While the linear space spanned by $\{v_1v_2 - q(v_1,v_2)| v_1, v_2 \in V\}$ is also of dimension $\binom{2d}{2}$. Therefore $\mathfrak{spin}(V,q)$ is generated by $\{v_1v_2 - q(v_1,v_2)| v_1, v_2 \in V\}$. $\square$

**Lemma 6.5.** *For any $v_1, v_2 \in (V,q)$, the infinitesimal representation $d\widetilde{Ad} : \mathfrak{spin}(V,q) \to$*

*$\mathfrak{so}(V,q)$ maps*

$$d\widetilde{Ad} : v_1 v_2 - q(v_1, v_2) \mapsto (v \mapsto 2q(v_2, v)v_1 - 2q(v_1, v)v_2) \,. \tag{6.20}$$

*Proof.* $d\widetilde{Ad}$ is derived by differentiating $\widetilde{Ad}$,

$$\begin{aligned}
&\lim_{t\to 0} \frac{\exp\left(\left(v_1 v_2 - q(v_1, v_2)\right) t\right) v \exp\left(-\left(v_1 v_2 - q(v_1, v_2)\right) t\right)}{t} \\
=&2q(v_2, v)v_1 - 2q(v_1, v)v_2.
\end{aligned}$$

□

Next we give expressions for the Lie algebras and the correspondence in terms of specific bases and work on the expressions.

**Lemma 6.6.** *Let $T \in GL(V,\mathbb{R})$, in terms of the basis $\left\{\partial_{x_j}, \partial_{y_j}\right\}_{j=1}^{d}$ of $(V,q)$, the matrix representation of $T$ is*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

*$T \in O(V,q)$ if and only if*

$$A^t A + C^t C = B^t B + C^t C = id_d, \quad A^t B = -C^t D, \tag{6.21}$$

*or*

$$AA^t + BB^t = CC^t + DD^t = id_d, \quad AC^t = -BD^t, \tag{6.22}$$

*where $id_d$ is the identity matrix of dimension $d \times d$.*

Let $T \in GL(V,\mathbb{R})$, with respective to the basis $\{\partial_{x_j}, \partial_{y_j}\}_{j=1}^d$, $T$ is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We extend $T$ complex linearly to $V_{\mathbb{C}}$ and express it using the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{j=1}^d$. The matrix representation is $\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$. The two matrix representations are similar, i.e.

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} = W \begin{pmatrix} A & B \\ C & D \end{pmatrix} W^{-1} \tag{6.23}$$

where

$$W = \begin{pmatrix} id_d & i\, id_d \\ id_d & -i\, id_d \end{pmatrix}, \quad W^{-1} = \frac{1}{2} \begin{pmatrix} id_d & id_d \\ -i\, id_d & i\, id_d \end{pmatrix},$$

and $id_d$ is the identity matrix of dimension $d \times d$, $P = (A + D - i(B - C))/2$ and $Q = (A - D + i(B + C))/2$.

**Lemma 6.7.** *Let $T \in GL(V,\mathbb{R}) \subset GL(V_{\mathbb{C}},\mathbb{C})$, in terms of the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{j=1}^d$ of $V_{\mathbb{C}}$,*

$$T = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}.$$

*All the following statements are equivalent*

1. $T \in O(V,q)$;

2. $\bar{Q}^t Q + P^t \bar{P} = id_d$ *and* $\bar{Q}^t P + P^t \bar{Q} = 0$;

3. $P\bar{P}^t + Q\bar{Q}^t = id_d$ *and* $PQ^t + QP^t = 0$

4. $T^t \begin{pmatrix} 0 & id_d \\ id_d & 0 \end{pmatrix} T = \begin{pmatrix} 0 & id_d \\ id_d & 0 \end{pmatrix}$

*where $id_d$ is the identity matrix of dimension $d \times d$.*

**Lemma 6.8.** *Let $T \in \mathfrak{gl}(V,\mathbb{R}) \subset \mathfrak{gl}(V_{\mathbb{C}},\mathbb{C})$, in terms of the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{j=1}^d$ of $V_{\mathbb{C}}$,*

$$T = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}.$$

*$T \in \mathfrak{o}(V,q)$ if and only if*

$$P^* = -P, \quad Q^t = -Q, \tag{6.24}$$

*or*

$$T^t \begin{pmatrix} 0 & id_d \\ id_d & 0 \end{pmatrix} + \begin{pmatrix} 0 & id_d \\ id_d & 0 \end{pmatrix} T = 0$$

*where $id_d$ is the identity matrix of dimension $d \times d$.*

Using the basis $\left\{\partial_{z_j}\partial_{z_k}, \partial_{z_j}\partial_{\bar{z}_k}, \partial_{\bar{z}_j}\partial_{\bar{z}_k}\right\}_{1\le j<k\le d}$ of $\mathfrak{spin}(V_{\mathbb{C}}, q_{\mathbb{C}})$ and and the basis

$\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{1\le j\le d}$ of $V_{\mathbb{C}}$, the infinitesimal representation $\widetilde{Ad}$,

$$d\widetilde{Ad}: \partial_{z_j}\partial_{z_k} \mapsto \begin{pmatrix} 0 & e_{jk} - e_{kj} \\ 0 & 0 \end{pmatrix},$$

$$\partial_{\bar{z}_j}\partial_{\bar{z}_k} \mapsto \begin{pmatrix} 0 & 0 \\ e_{jk} - e_{kj} & 0 \end{pmatrix},$$

$$\partial_{z_j}\partial_{\bar{z}_k} - \frac{1}{2}\delta_{jk} \mapsto \begin{pmatrix} e_{jk} & 0 \\ 0 & -e_{kj} \end{pmatrix},$$

where $e_{jk}$ is the $d \times d$ matrix which is 1 at the entry in $j$-th row and $k$-th column and 0 elsewhere. Putting above expressions into a concise form over $\mathfrak{spin}(V, q)$

$$d\widetilde{Ad}: \quad \frac{1}{2}\begin{pmatrix} \partial_z^t & \partial_{\bar{z}}^t \end{pmatrix} \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \end{pmatrix} \mapsto \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \tag{6.25}$$

where $\partial_z = (\partial_{z_1}, \dots, \partial_{z_d})^t$, $\partial_{\bar{z}} = \overline{\partial_z}$ and $\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$ satisfies (6.24). Similarly, the infinitesimal action $d\mu$,

$$d\mu: \quad \mathfrak{spin}(V, q) \to \mathfrak{u}(\wedge^* V) \tag{6.26}$$

$$\frac{1}{2}\begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \end{pmatrix}^t \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} a^\dagger \\ a \end{pmatrix}^t \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^\dagger \\ a \end{pmatrix}, \tag{6.27}$$

where

$$a^\dagger = \left(a_1^\dagger, a_2^\dagger, \dots, a_d^\dagger\right)^t, \quad a = (a_1, a_2, \dots, a_d)^t.$$

Observe that with respective to the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{1 \le j \le d}$,

$$\begin{pmatrix} P & 0 \\ 0 & \bar{P} \end{pmatrix} \in O(V,q) \text{ if and only if } P \in U(d),$$

which yields an embedding of $U(d)$ into $O(V,q)$. According to (6.25), elements in $\widetilde{Ad}^{-1}(U(d))$ can be written as

$$\exp\left(\sum_{1 \le j,k \le d} a_{jk}\left(\partial_{z_j}\partial_{\bar{z}_k} - \frac{1}{2}\delta_{jk}\right)\right), \quad (a_{jk}) \in \mathfrak{u}(d).$$

Applying the action of $\widetilde{Ad}^{-1}(U(d))$ to the $|0\rangle$ of $\bigwedge^*(V, \langle\cdot,\cdot\rangle_V)$

$$\begin{aligned} &\mu\left(\exp\left(\sum_{1 \le j,k \le d} a_{jk}\left(\partial_{z_j}\partial_{\bar{z}_k} - \frac{1}{2}\delta_{jk}\right)\right)\right)|0\rangle \\ &= \exp\left(-\frac{1}{2}\sum_{1 \le j \le d} a_{jj}\right)|0\rangle \\ &= \exp\left(-\frac{1}{2}\mathrm{Tr}\left((a_{jk})\right)\right)|0\rangle \quad (a_{jk}) \in \mathfrak{u}(d). \end{aligned}$$

Therefore the stabilizer of $|0\rangle$ under the $Spin(V,q)$ action contains

$$\left\{\exp\left(\sum_{1 \le j,k \le n} a_{jk}\left(\partial_{z_j}\partial_{\bar{z}_k} - \frac{1}{2}\delta_{jk}\right)\right)\middle|\, \mathrm{Tr}\left((a_{jk})\right) = 0,\ (a_{jk}) \in \mathfrak{u}(d)\right\}. \tag{6.28}$$

We will show that the stabilizer of $|0\rangle$ is exactly this set.

In the last part of this section, we discuss the relation between the Pin group representation and pure quasi-free states. Let $\omega$ be state of $\mathcal{F}_a$, i.e. a semi-definite linear transform with trace 1. Correlation functions associated to the state $\omega$ are defined as

$$(x_1, x_2, \ldots, x_k) \mapsto \mathrm{Tr}_{\mathcal{F}_a}\left(\left(x_1 a_{j_1}^{\#}\right)\left(x_2 a_{j_2}^{\#}\right)\cdots\left(x_k a_{j_k}^{\#}\right)\omega\right) \tag{6.29}$$

where $x_l \in \mathbb{C}$, $1 \leq j_l \leq d, k \in \mathbb{N}$ and $\#$ means that it is either an annihilation operator or a creation operator. For simplicity,

$$\left\langle a_{j_1}^{\#} a_{j_2}^{\#} \cdots a_{j_k}^{\#} \right\rangle_{\omega} \coloneqq \mathrm{Tr}_{\mathcal{F}_a}\left(\omega, a_{j_1}^{\#} a_{j_2}^{\#} \cdots a_{j_k}^{\#} \omega\right), \tag{6.30}$$

where $1 \leq j_l \leq d,\ k \in \mathbb{N}$. Claim that the collection of correlation functions determines the Fock state up to phases. To prove the claim, through the Clifford action $\mu$, the collection of all correlation functions amounts to a function defined on $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$

$$f_{\omega} : T \in Cl(V_{\mathbb{C}}, q_{\mathbb{C}}) \mapsto \mathrm{Tr}_{\mathcal{F}_a}\left(\mu(T)\omega\right). \tag{6.31}$$

$\bigwedge^* V$ can be identified as $\mathbb{C}^{2^d}$ with inner metric $\tilde{q}$, since $Cl(V_{\mathbb{C}}, q_{\mathbb{C}})$ is isomorphic to the matrix algebra $Mat(2^d, \mathbb{C})$ and $\bigwedge^* V$ is its finite dimensional irreducible representation. If $f_{\omega}$ vanishes, one can find $T \in Mat(2^d, \mathbb{C})$ such that $\mu(T) = \omega^*$, then $\mathrm{Tr}_{\mathcal{F}_a}\left(\omega^* \omega\right) = 0$ implies that $\omega = 0$.

The state $\omega$ is quasi-free if

$$\left\langle a^{\#}_{j_1} a^{\#}_{j_2} \cdots a^{\#}_{j_{2n+1}} \right\rangle_{\omega} = 0, \tag{6.32}$$

$$\left\langle a^{\#}_{j_1} a^{\#}_{j_2} \cdots a^{\#}_{j_{2n}} \right\rangle_{\omega} = \sum_{\sigma \in S_{ad}} \operatorname{sgn}(\sigma) \left\langle a^{\#}_{j_{\sigma(1)}} a^{\#}_{j_{\sigma(2)}} \right\rangle_{\omega} \cdots \left\langle a^{\#}_{j_{\sigma(2n-1)}} a^{\#}_{j_{\sigma(2n)}} \right\rangle_{\omega}, \tag{6.33}$$

where $\operatorname{sgn}(\sigma)$ denotes the sign of permutation $\sigma$ and $S_{ad}$ is a subset of the permutation group $S_{2n}$ such that

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1), \quad \sigma(2k-1) < \sigma(2k).$$

Another characterization of $\omega$ being quasi-free is given based on the generalized one-particle density matrix, which is defined

**Definition 6.9.** Let $\omega$ be a state of $\bigwedge^* (V, \langle \cdot, \cdot \rangle_V)$, the associated generalized one-particle density matrix $S_\omega$ is a complex linear transform on $V_{\mathbb{C}}$ such that

$$2q_{\mathbb{C}} \left( S_\omega u, v \right) = \operatorname{Tr}_{\mathcal{F}_a} \left( \mu(u) \mu(v) \omega \right), \tag{6.34}$$

or equivalently $\langle v, S_\omega u \rangle_{V_{\mathbb{C}}} = \operatorname{Tr}_{\mathcal{F}_a} \left( \mu(u) \mu^*(v) \omega \right)$, where $u, v \in V_{\mathbb{C}}$.

$\omega$ is quasi-free if and only if the generalized one-particle density matrix $S_\omega$ satisfies

$$1 \geq S_\omega^* = S_\omega \geq 0, \quad S_\omega^2 = S_\omega. \tag{6.35}$$

Let $\omega = id$, the generalized one-particle density matrix $S_{id}$ is the identity map on the subspace spanned by $\{\partial_{\bar{z}_j}\}_{j=1}^d$. For short, when $\omega$ is pure, we may use associated the Fock state to denote $\omega$.

Let $T \in Pin(V,q)$ and $|0\rangle$ be the vacuum of $\bigwedge^* (V, \langle\cdot,\cdot\rangle_V)$, the generalized one particle matrix $S_{\mu(T^{-1})|0\rangle}$ for $\mu(T^{-1})|0\rangle$ is then $\widetilde{Ad}_T^* S_{id} \widetilde{Ad}_T$. Specifically, in terms of the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{j=1}^d$, if

$$\widetilde{Ad}_T = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix},$$

then

$$S_{\mu(T^{-1})|0\rangle} = \begin{pmatrix} Q^t\bar{Q} & Q^t\bar{P} \\ P^t\bar{Q} & P^t\bar{P} \end{pmatrix}.$$

Besides, regarding $a^\dagger$ and $a$ as row vectors, two-particle correlation functions are

$$a^\dagger a: \quad g^t\Gamma f := \left\langle \mu(T^{-1})|0\rangle, (a^\dagger f^t)(ag^t)\mu(T^{-1})|0\rangle \right\rangle_{\mathcal{F}_a} = fQ^*Qg^t,$$

$$aa: \quad g^t\Lambda f := \left\langle \mu(T^{-1})|0\rangle, (af^t)(ag^t)\mu(T^{-1})|0\rangle \right\rangle_{\mathcal{F}_a} = fP^*Qg^t,$$

where $f,g \in \mathbb{C}^{2^d}$. For short,

$$\langle a^\dagger a\rangle_{\mu(T^{-1})|0\rangle} = \Gamma = Q^t\bar{Q}, \quad \langle aa\rangle_{\mu(T^{-1})|0\rangle} = \Gamma = Q^t\bar{P}$$

then $S_{\mu(T^{-1})|0\rangle}$

$$S_{\mu(T^{-1})|0\rangle} = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & id_d - \bar{\Gamma} \end{pmatrix}.$$

Finally we give the characterization of pure quasi-free states by the Clifford action $\mu$ of $Pin(V,q)$

**Theorem 6.10.** *Let $\omega$ be a pure state on $\bigwedge^* (V, \langle\cdot,\cdot\rangle_V)$. $\omega$ is quasi-free if and only if there is $T \in Pin(V,q)$, such that $\omega = |\mu(T^{-1})\,|0\rangle\rangle\,\langle\mu(T^{-1})\,|0\rangle|$, where $|0\rangle = 1$ is the vacuum of $\mathcal{F}_a$.*

*Proof.* The "if" part follows from the above discussion. If there is $T \in Pin(V,q)$ and $\omega = \mu(T^{-1})\,|0\rangle$, then the generalized one-particle density matrix is $S_T = \widetilde{Ad}_T^* S_{id} \widetilde{Ad}_T$ and it satisfies Condition (6.35).

To show the "only if" part, we need to construct a $\widetilde{Ad}_T$ from the generalized one-particle density matrix

$$S_\omega = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & 1 - \bar{\Gamma} \end{pmatrix},$$

where $\Gamma^* = \Gamma$, $\Lambda^t = -\Lambda$ and $S_\omega^2 = S_\omega$. Observe that

$$\left(S_\omega - \frac{1}{2}\right)^2 = \frac{1}{4},$$

which implies eigenvalues of $S_\omega - 1/2$ are either $1/2$ or $-1/2$. Since

$$\mathrm{Tr}(S_\omega - 1/2) = \mathrm{Tr}(\Gamma) + \mathrm{Tr}(1 - \bar{\Gamma}) - d = \mathrm{Tr}(\Gamma) - \mathrm{Tr}(\Gamma^t) = 0,$$

the multiplicity of $1/2$ is the same as the multiplicity of $-1/2$. Let $V_{1/2}$ and $V_{-1/2}$ denote the eigenspace associated to $1/2$ and $-1/2$ respectively. Suppose $u$ is an eigenvector for $1/2$, then

$$\overline{\left(S_\omega - \frac{1}{2}\right) u} = -\left(S_\omega - \frac{1}{2}\right) \bar{u} = \frac{1}{2}\bar{u}.$$

It means that the complex conjugation is an isomorphism from $V_{1/2}$ to $V_{-1/2}$. Therefore there is a unitary matrix $U$ on $V_{\mathbb{C}}$ such that $U \in O(V,q)$ and

$$S_\omega = U \begin{pmatrix} -\frac{1}{2}id & 0 \\ 0 & +\frac{1}{2}id \end{pmatrix} U^* + \frac{1}{2} = U \begin{pmatrix} 0 & 0 \\ 0 & id \end{pmatrix} U^*$$

where $id$ is the $d \times d$ identity matrix. Choosing $\widetilde{Ad}_{\tilde{T}} = U^*$ and $\check{T} \in \widetilde{Ad}^{-1}\left(\tilde{T}\right)$, $\omega$ and $\mu\left(\check{T}\right)\Omega$ have the same generalized one-particle density matrix. Since both states are quasi-free states, all correlation functions of them coincide. The collection of correlation functions determines a state up to a phase. Therefore there is $\theta \in \mathbb{R}$ such that $\omega = e^{\theta i}\mu\left(\check{T}\right)\Omega$ and $T$ can be chosen as

$$T = \exp\left(-\frac{2\theta i}{d} \sum_{1 \le j,k \le d} \left(\partial_{z_j}\partial_{\bar{z}_k} - \frac{1}{2}\delta_{jk}\right)\right)\check{T}.$$

$\square$

At this moment, we are able to show the stabilizer of $\Omega$ under the $Pin(V,q)$ action. Suppose $T \in Pin(V,q)$ and $\mu(T^{-1})\Omega = \Omega$, then $S_{\mu(T^{-1})\Omega} = S_\Omega$. With respect to the basis $\{\partial_{z_j}, \partial_{\bar{z}_j}\}_{j=1}^n$,

$$\widetilde{Ad}_T = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix},$$

then $Q^t\bar{Q} = 0$ and $P^t\bar{P} = id$. Thus $\widetilde{Ad}_T \in U(d)$ and the stabilizer of $\Omega$ is (6.28).

**Example 6.11.** Let $d = 1$, i.e. $\dim_{\mathbb{R}}(V) = 2$. $\{\partial_x, \partial_y\}$ is the canonical basis of $(V,q)$ and $q = dx \otimes dx + dy \otimes dy$. Then the canonical basis of $Cl(V,q)$ is $\{1, \partial_x, \partial_y, \partial_x\partial_y\}$

and

$$\partial_x^2 = 1,\ \partial_y^2 = 1,\ \partial_x\partial_y = -\partial_y\partial_x.$$

The real algebra $Cl(V,q)$ is isomorphic to the matrix algebra $Mat(\mathbb{R},2)$ and the isomorphism is given by

$$1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial_x \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \partial_y \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this case, $Pin(V,q)$ is generated by $\{\cos\theta\partial_x + \sin\theta\partial_y | \theta \in [0,2\pi]\}$ and

$$Spin(V,q) = \{\cos\theta + \sin\theta\partial_x\partial_y \,|\, \theta \in [0,2\pi]\}.$$

The multiplication law in $Pin(V,q)$ is

$$(\cos\theta_1\partial_x + \sin\theta_1\partial_y)(\cos\theta_2\partial_x + \sin\theta_2\partial_y) = \cos(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1)\partial_x\partial_y.$$

The explicit expression for the skewed adjoint representation $\widetilde{Ad}$, for $x, y \in \mathbb{R}$,

$$\widetilde{Ad}_{\cos\theta\partial_x + \sin\theta\partial_y}(x\partial_x + y\partial_y) = -\left((x\cos 2\theta + y\sin 2\theta)\partial_x + (x\sin 2\theta - y\cos 2\theta)\partial_y\right),$$

$$\widetilde{Ad}_{\cos\theta + \sin\theta\partial_x\partial_y}(x\partial_x + y\partial_y) = (x\cos 2\theta + y\sin 2\theta)\,\partial_x + (y\cos 2\theta - x\sin 2\theta)\,\partial_y.$$

With respect to the basis $\{\partial_z, \partial_{\bar{z}}\}$ of $V_{\mathbb{C}}$,

$$\widetilde{Ad} : \cos\theta\partial_x + \sin\theta\partial_y = e^{\theta i}\partial_z + e^{-\theta i}\partial_{\bar{z}} \mapsto -\begin{pmatrix} 0 & e^{2\theta i} \\ e^{-2\theta i} & 0 \end{pmatrix}$$

$$\cos\theta + \sin\theta\partial_x\partial_y = e^{\theta i} - 2i\sin\theta\partial_z\partial_{\bar{z}} \mapsto \begin{pmatrix} e^{-2\theta i} & 0 \\ 0 & e^{2\theta i} \end{pmatrix}.$$

To compute the infinitesimal representation $d\widetilde{Ad}$, notice that $\cos\theta + \sin\theta\partial_x\partial_y$ is generated by $\theta\partial_x\partial_y = i\theta(1 - 2\partial_z\partial_{\bar{z}})$, i.e. $\exp(\theta\partial_x\partial_y) = \cos\theta + \sin\theta\partial_x\partial_y$. Then with respect to the basis $\{\partial_z, \partial_{\bar{z}}\}$ of $V_{\mathbb{C}}$,

$$d\widetilde{Ad} : i\theta(1 - 2\partial_z\partial_{\bar{z}}) = \frac{1}{2}(\partial_z \ \partial_{\bar{z}})\begin{pmatrix} -2\theta i & 0 \\ 0 & 2\theta i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \end{pmatrix} \mapsto \begin{pmatrix} -2\theta i & 0 \\ 0 & 2\theta i \end{pmatrix}.$$

There are essentially two pure quasi-free states

$$\left(e^{\theta i}\partial_z + e^{-\theta i}\partial_{\bar{z}}\right)\Omega = e^{\theta i}\partial_z \quad \text{and} \quad \left(e^{\theta i} - 2i\sin\theta\partial_z\partial_{\bar{z}}\right)\Omega = e^{\theta i}.$$

Furthermore, to find all quasi-free states, consider all candidates

$$\cos^2(\theta)\left|a + b\partial_z\right\rangle\left\langle a + b\partial_z\right| + \sin^2(\theta)\left|c + d\partial_z\right\rangle\left\langle c + d\partial_z\right|,$$

where $|a|^2 + |b|^2 = 1$, $|c|^2 + |d|^2 = 1$ and $a\bar{c} + b\bar{d} = 0$. Testing the candidates for conditions

(6.32)(6.33), we only need to consider $\left\langle a_{j1}^{\#}\right\rangle_{\omega}$, which amounts to the equation

$$\cos^2\theta a\bar{b} + \sin^2\theta c\bar{d} = 0.$$

The equation has only one family of solutions: $a = d = 1, b = d = 0, \theta \in [0, \pi/2]$. To show the uniqueness, based on conditions $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$, $a, b, c, d$ are expressed as

$$a = \cos\varphi_1 e^{\theta_a i},\ \ b = \sin\varphi_1 e^{\theta_b i},\ \ c = \cos\varphi_2 e^{\theta_c i},\ \ d = \sin\varphi_2 e^{\theta_d i}.$$

Using the condition $a\bar{c} + b\bar{d} = 0$, we obtain

$$\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 e^{(\theta_c+\theta_b-\theta_a-\theta_d)i} = 0.$$

It leads to two cases

1. $e^{(\theta_c+\theta_b-\theta_a-\theta_d)i} = 1$, $\varphi_1 - \varphi_2 = \frac{\pi}{2} + k\pi$ or $e^{(\theta_c+\theta_b-\theta_a-\theta_d)i} = -1$, $\varphi_1 + \varphi_2 = \frac{\pi}{2} + k\pi$.

2. $e^{(\theta_c+\theta_b-\theta_a-\theta_d)i} \neq \pm 1$, $\sin\varphi_1\sin\varphi_2 = 0$ and $\cos\varphi_1\cos\varphi_2 = 0$.

Case 2. is contained in the solutions with $a = d = 1, b = d = 0$. Combining Case 1. with the equation $\cos^2\theta a\bar{b} + \sin^2\theta c\bar{d} = 0$, we still obtain the solutions with $a = d = 1, b = d = 0$. Therefore all quasi-free states are

$$\cos^2\theta \left|1\right\rangle\left\langle 1\right| + \sin^2\theta \left|\partial_z\right\rangle\left\langle\partial_z\right|.$$

### 6.2.2 Abstract Theory

Let $\mathcal{H}$ be a real Hilbert space endowed with inner product $q$ and compatible complex structure $J$, i.e. for $u, v \in \mathcal{H}$

$$J^2 u = -u, \quad q(Ju, Jv) = q(u, v).$$

Then $\mathcal{H}$ can be viewed as a complex Hilbert space with the following Hermitian form

$$\langle u, v\rangle_{\mathcal{H}} := q(u, v) + i\, q(Ju, v), \quad \text{for any } u, v \in \mathcal{H}. \tag{6.36}$$

To distinguish which structure is used, we denote the space and its bilinear form together, i.e. using $(\mathcal{H}, q)$ for the underlying real structure and using $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ for the complex structure.

Complexify $\mathcal{H}$: $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{H}$ and extend $q$ complex linearly: $q_{\mathbb{C}} = id \otimes q$. The Clifford algebra $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ is defined as

$$Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}}) := \bigoplus_{n\geq 0} \mathcal{H}_{\mathbb{C}}^{\otimes^n} / \mathcal{I}\left(\{u \otimes u - q_{\mathbb{C}}(u) | u \in \mathcal{H}_{\mathbb{C}}\}\right) \tag{6.37}$$

where $q_{\mathbb{C}}(u) := q_{\mathbb{C}}(u, u)$ and $\mathcal{I}\left(\{u \otimes u - q_{\mathbb{C}}(u) | u \in \mathcal{H}_{\mathbb{C}}\}\right)$ denotes the ideal generated by elements in the form $u \otimes u - q_{\mathbb{C}}(u)$. The complex conjugation on $\mathcal{H}_{\mathbb{C}}$ is $\overline{c \otimes u} = \bar{c} \otimes u$, where $c \in \mathbb{C}$ and $u \in (\mathcal{H}, q)$, and it is extended linearly to $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$. The transpose is defined as

$$(u_1 u_2 \ldots u_n)^t := u_n u_{n-1} \ldots u_2 u_1, \tag{6.38}$$

where $u_j \in \mathcal{H}_{\mathbb{C}}$. We also define a Hermitian form on $\mathcal{H}_{\mathbb{C}}$, for $u, v \in \mathcal{H}_{\mathbb{C}}$,

$$\langle u, v \rangle_{\mathcal{H}_{\mathbb{C}}} := 2q_{\mathbb{C}}(\bar{u}, v). \tag{6.39}$$

Using the complex conjugation and the transpose, an $*$ structure on $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ is

$$u^* := \bar{u}^t \tag{6.40}$$

where $u \in Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$. $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ can also be endowed with a maximum $C^*$-norm $\|\cdot\|_{C^*}$ [SS64] and its $C^*$-completion is denoted by $\overline{Cl}(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$.

Any element in $\mathcal{H}_{\mathbb{C}}$ can be decomposed as a sum of terms in the forms

$$\frac{u - iJu}{2}, \quad \frac{u + iJu}{2}, \quad u \in (\mathcal{H}, q). \tag{6.41}$$

Each form yields an identification of $(H, \langle \cdot, \cdot \rangle_{\mathcal{H}})$:

$$\frac{u - iJu}{2} \to u$$

is complex linear and

$$\frac{u + iJu}{2} \to u$$

is complex conjugate linear. Let $\mathcal{H}_{\mathbb{C}}^{1,0}$ denote the subspace spanned by $\frac{u-iJu}{2}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$ denote the subspace spanned by $\frac{u+iJu}{2}$. Then

$$\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{1,0} \bigoplus \mathcal{H}_{\mathbb{C}}^{0,1} \quad \text{and} \quad \overline{\mathcal{H}_{\mathbb{C}}^{1,0}} = \mathcal{H}_{\mathbb{C}}^{0,1}$$

where the bar means the complex conjugation over $\mathcal{H}_{\mathbb{C}}$.

**Definition 6.12.** Let $T$ be an operator on $\mathcal{H}_{\mathbb{C}}$, the complex conjugation $\bar{T}$ of $T$ is defined as

$$\bar{T}u := \overline{T\bar{u}}, \quad u \in \mathcal{H}_{\mathbb{C}}. \tag{6.42}$$

$T$ is real if $\bar{T} = T$.

If $\bar{T} = T$, for any $u \in (\mathcal{H}, q)$, $\overline{Tu} = \bar{T}\bar{u} = Tu$, i.e. $Tu$ is real and $Tu \in (\mathcal{H}, q)$. The space $(\mathcal{H}, q)$ is an invariant subspace of $T$. Therefore $T$ is of the form $id \otimes T|_{(\mathcal{H},q)}$.

In the abstract setting, all the constructions are defined as word-to-word translation of the finite-dimensional case except that we will take care of two different topology: $C^*$ topology and the strong topology.

$$\begin{array}{ccc} Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}}) & \xrightarrow{\widetilde{Ad}} & GL\left(Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})\right) \\ \big\downarrow \mu & & \\ \wedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}) & & \end{array}$$

where we abuse the notation and $\wedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ means the norm completion of $\bigoplus_{n\geq 0} \wedge^n(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$. $\wedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is the Fock space $\mathcal{F}_a$ defined in Section 6.1. The representation $\mu$ is first defined on the two subspaces $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$

$$\mu : \frac{u - iJu}{2} \mapsto a^{\dagger}(u), \quad \frac{u + iJu}{2} \mapsto a(u), \quad u \in (\mathcal{H}, q), \tag{6.43}$$

where $a^{\dagger}(u)$ and $a(u)$ denote creation and annihilation operators, and it is extended to $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ using the universal property of Clifford algebras. $Pin(\mathcal{H}, q)$ is a group generated by

$$\{u \in (\mathcal{H}, q) | q(u) = 1\},$$

and its $C^*$-completion is denoted by $\overline{Pin}(\mathcal{H}, q)$.

**Proposition 6.13.** *$\mu$ defines a $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$-module structure on $\bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and it satisfies*

1. $\mu(u)^* = \mu(u^*), \quad u \in Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$.

2. $\mu : Pin(\mathcal{H}, q) \to U(\bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}))$.

*where $U(\bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}))$ denotes the unitary group of $\bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$.*

*Proof.* To extend the definition of $\mu$, since $\mathcal{H}_{\mathbb{C}}$ is a direct sum of $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$, if $\mu$ is complex linear on the two subspaces, then it can be extended to a complex linear map on $\mathcal{H}_{\mathbb{C}}$. Furthermore, if $\mu$ satisfies

$$\mu(u)\mu(v) + \mu(v)\mu(u) = 2q_{\mathbb{C}}(u, v), \quad u, v \in \mathcal{H}_{\mathbb{C}}, \tag{6.44}$$

by the universality of Clifford algebra, $\mu$ extends to $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$. Thus $\bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$-module.

To verify $\mu$ is complex linear on the two subspaces, it suffices to check for

$u \in (\mathcal{H}, q)$ and $\Psi \in \bigwedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$,

$$
\begin{aligned}
&\mu\left(i\left(\frac{u - iJu}{2}\right)\right)\Psi \\
=&\mu\left(\frac{Ju - iJJu}{2}\right)\Psi = (Ju) \wedge \Psi = iu\Psi \\
=&i\mu\left(\frac{u - iJu}{2}\right)\Psi,
\end{aligned}
$$

and suppose $\Psi = \psi_1 \wedge \cdots \wedge \psi_n$ for $\psi \in (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$,

$$
\begin{aligned}
&\mu\left(i\left(\frac{u + iJu}{2}\right)\right)\Psi \\
=&\mu\left(\frac{-Ju + iJ(-Ju)}{2}\right)\Psi \\
=&\sum_{j=1}^{n}(-1)^{j+1}\langle -Ju, \psi_j\rangle\, \psi_1 \wedge \cdots \wedge \hat{\psi}_j \wedge \cdots \wedge \psi_n \\
=&i\sum_{j=1}^{n}(-1)^{j+1}\langle u, \psi_j\rangle\, \psi_1 \wedge \cdots \wedge \hat{\psi}_j \wedge \cdots \wedge \psi_n \\
=&i\mu\left(\frac{u + iJu}{2}\right)\Psi.
\end{aligned}
$$

To show that $\mu$ satisfies identity (6.44) for $\mathcal{H}_{\mathbb{C}}$, we check all combinations of elements from $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$. For example, for $u, v \in (\mathcal{H}, q)$,

$$
\mu\left(\frac{u - iJu}{2}\right)\mu\left(\frac{v + iJv}{2}\right) + \mu\left(\frac{v + iJv}{2}\right)\mu\left(\frac{u - iJu}{2}\right) = \langle v, u\rangle_{\mathcal{H}}
$$

and

$$
q_{\mathbb{C}}\left(\frac{u - iJu}{2}, \frac{v + iJv}{2}\right) = \frac{1}{2}\left(q(v,u) + iq(Jv,u)\right) = \frac{1}{2}\langle v, u\rangle_{\mathcal{H}}.
$$

Other cases are computed similarly.

To show Property (1), it suffices to consider $u \in (\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$, since $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ is generated by $(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ and $\mu$ defines a Clifford module action. It further reduces to, for $u \in (\mathcal{H}, q)$

$$\mu^*\left(\frac{u - iJu}{2}\right) = \left(a^\dagger(u)\right)^* = a(u) = \mu\left(\frac{u + iJu}{2}\right).$$

To show Property (2), it suffices to consider generators $u \in (\mathcal{H}, q)$ such that $q(u) = 1$. For any $v \in (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$,

$$\langle \mu(u)v, \mu(u)v\rangle_{\mathcal{H}} = \langle \mu^*(u)\mu(u)v, v\rangle_{\mathcal{H}} = \langle \mu(\bar{u})\mu(u)v, v\rangle_{\mathcal{H}} = \left\langle \mu(u^2)v, v\right\rangle_{\mathcal{H}} = \langle v, v\rangle_{\mathcal{H}}.$$

□

**Proposition 6.14.** *The image of* $\widetilde{Ad} : \overline{Pin}(\mathcal{H}, q) \to U\left(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}\right)$ *contains the subset*

$$\left\{ id + T \,\middle|\, T \in \mathcal{L}^1(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}) \quad and \quad \bar{T} = T \right\},$$

where $id$ is the identity map on $(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$.

*Proof.* Given an operator $id + T \in U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$, where $T \in \mathcal{L}^1(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$, we will approximate it by $\widetilde{Ad}_{(h_n)}$ where $h_n \in Pin(\mathcal{H}, q)$, such that as $n \to \infty$, $h_n \to h$ and

$$\widetilde{Ad}_{(h_n)} \xrightarrow{\mathcal{L}^1(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})} id + T.$$

The idea is to construct finite-rank truncation $id + T_n$ of $id + T$, and then realize $id + T_n$ through $\widetilde{Ad}$ by some $h_n \in Pin(\mathcal{H}, q)$.

Since $id + T \in U(\mathcal{H}_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}})$ and $T$ is compact, $T$ is normal and diagonalizable. Denote eigenvalues of $T$ by $\lambda_j$, $j \in \mathbb{N}$. They satisfy $|1 + \lambda_j| = 1$. Without loss of generality, assume that

$$|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \ldots, \quad \text{and} \quad \lambda_0 = -2.$$

Because $T$ is real, a complex value and its complex conjugate appear as a pair as eigenvalues of $T$. We further suppose $\lambda_{2j} = \bar{\lambda}_{2j-1}$ for $j > 0$, $m_j$ is the multiplicity of $\lambda_{2j}$ and

$$\lambda_{2j} = e^{-2\theta_j i} - 1 \quad \lambda_{2j-1} = e^{2\theta_j i} - 1.$$

The eigenvectors associated to $\lambda_{2j}$ and $\bar{\lambda}_{2j-1}$ may not belong to $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$ respectively. This statement is true for all $j > 0$ if and only if $T$ commutes with the complex structure on $J$ on $\mathcal{H}$. Let $V_{n\mathbb{C}}$ denote the complex subspace of $\mathcal{H}_{\mathbb{C}}$ spanned by eigenvectors corresponding to $\lambda_0, \lambda_1, \ldots, \lambda_{2n}$. $V_{n\mathbb{C}}$ is invariant under complex conjugation. Then the finite-rank truncation $id + T_n$ is defined as

$$id + T_n := id + T|_{V_{n\mathbb{C}}}.$$

Next we show

1. $id + T_n \in U(\mathcal{H}_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}})$ and $T_n$ is real;

2. $id + T_n$ is realized by $h_n$.

Let $u \in (\mathcal{H}_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}})$ and it is decomposed as $u = v + v^{\perp}$, where $v \in V_{n\mathbb{C}}$, $v^{\perp} \in V_{n\mathbb{C}}^{\perp}$ and

$V_{n\mathbb{C}}^{\perp}$ denotes the orthogonal complement of $V_{n\mathbb{C}}$. Then

$$
\begin{aligned}
&\langle (id+T_n)u, (id+T_n)u\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&= \langle (id+T_n)v, (id+T_n)v\rangle_{\mathcal{H}_{\mathbb{C}}} + \langle (id+T_n)v^{\perp}, (id+T_n)v^{\perp}\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&\quad + \langle (id+T_n)v, (id+T_n)v^{\perp}\rangle_{\mathcal{H}_{\mathbb{C}}} + \langle (id+T_n)v^{\perp}, (id+T_n)v\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&= \langle (id+T)v, (id+T)v\rangle_{\mathcal{H}_{\mathbb{C}}} + \langle v^{\perp}, v^{\perp}\rangle_{\mathcal{H}_{\mathbb{C}}} + \langle (id+T)v, v^{\perp}\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&\quad + \langle v^{\perp}, (id+T)v\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&= \langle v, v\rangle_{\mathcal{H}_{\mathbb{C}}} + \langle v^{\perp}, v^{\perp}\rangle_{\mathcal{H}_{\mathbb{C}}}.
\end{aligned}
$$

That $T_n$ is real follows from the observation that $V_{n\mathbb{C}}$ is invariant under complex conjugation. Let $V_n = V_{n\mathbb{C}} \cap (\mathcal{H}, q)$. $(V_n, q)$ is finite-dimensional real Hilbert space. Since $T_n$ is real and recall the definition of $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}$, $V_n$ is invariant under $T_n$ and $id + T_n|_{V_n} \in O(V_n, q)$. By Theorem 6.3, $\widetilde{Ad} : Pin(V_n, q) \to O(V_n, q)$ is a double covering. Then there is $h_n \in Pin(V_n, q) \subset Cl(V_n, q)$ such that

$$id + T_n|_{V_{n\mathbb{C}}} = \widetilde{Ad}_{(h_n)}.$$

$id + T_n$ and $\widetilde{Ad}_{(h_n)}$ also coincide on $(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$, because $\widetilde{Ad}_{(h_n)}$ is an identity map on $V_{n\mathbb{C}}^{\perp}$. However in order to show the convergence of $h_n$, we will construct $h_n$ inductively and explicitly. The construction of $h_0$ is not important and we use the covering map to find a candidate. Suppose $h_{n-1}$ is constructed and consider a pair of eigenvalues $\lambda_{2n}$ and $\lambda_{2n-1}$, and corresponding orthonormal eigenvector pair

$$(u_{n_1}, \bar{u}_{n_1}), \dots, (u_{n_{m_n}}, \bar{u}_{n_{m_n}})$$

and denote

$$x_{n_l} = \frac{u_{n_l} + \bar{u}_{n_l}}{2}, \quad y_{n_l} = i\frac{u_{n_l} - \bar{u}_{n_l}}{2}$$

where $l = n_1, \dots, n_{m_n}$. Based on the computation of Example 6.11, $T|_{(u_{n_l}, \bar{u}_{n_l})}$ corresponds to

$$\cos\theta_n + \sin\theta_n x_{n_l} y_{n_l}$$

and its infinitesimal generator is $\theta_n x_{n_l} y_{n_l}$. Since $x_{n_l} y_{n_l}$ commutes with each other,

$$h_n = h_{n-1} \exp\left(\sum_{l=n_1}^{n_{m_n}} \theta_n x_l y_l\right) = h_0 \exp\left(\sum_{j=1}^{n} \sum_{l=j_1}^{j_{m_j}} \theta_j x_l y_l\right)$$

The $C^*$-norm of $\cos\theta_n + \sin\theta_n x_{n_l} y_{n_l} - id$ is

$$\|\cos\theta_n + \sin\theta_n x_{n_l} y_{n_l} - id\|_{C^*} = \left|\sin\left(\frac{\theta_n}{2}\right)\right|$$

Therefore $h_n$ converges if

$$\sum_{j=1}^{n} m_j \left|\sin\left(\frac{\theta_j}{2}\right)\right|$$

has a limit. Note that the trace norm of $T$ is

$$\|T\|_{\mathcal{L}^1(\mathcal{H}_{\mathbb{C}})} = 2m_0 + 2\sum_{j=1}^{\infty} m_j |\lambda_{2j}| = 2m_0 + 4\sum_{j=1}^{\infty} m_j |\sin(\theta_j)|$$

Therefore $h_n$ converges and the sequence of infinitesimal generators also converges. Due to the construction of $\widetilde{Ad}_{(h_n)}$, $\widetilde{Ad}_{(h_n)} - id$ converges to $T$ in $\mathcal{L}^1(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$. $\square$

The inverse of Proposition 6.14 is also true, see [SS65, Corollary] [Ara71, Theorem 5.][1]. The images are called inner Bogoliubov transforms. If we consider the strong topology limits of the finite approximations in Proposition 6.14 under the representation $\mu$, they correspond to elements $T$ in $U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$ such that $\bar{T} = T$,

$$P_{\mathcal{H}_{\mathbb{C}}^{1,0}} T P_{\mathcal{H}_{\mathbb{C}}^{0,1}} \text{ is Hilbert-Schmidt,} \tag{6.45}$$

where $P_{\mathcal{H}_{\mathbb{C}}^{1,0}}$ and $P_{\mathcal{H}_{\mathbb{C}}^{0,1}}$ are projections onto $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$ respectively. The condition is called the Shale-Stinespring condition [SS65, Theorem] and the elements $T$ are called unitary implementable Bogoliubov transformations. Specifically in terms of $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$ and $\wedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$, since $T$ is in $U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$ and it is real, it can lift to an automorphism over $Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$. The existence of a unitary implementation, i.e. a unitary realization $\pi(T) \in U(\wedge^*(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}))$ which satisfies the adjoint relation

$$\pi(T)\mu(u)\pi^*(T) = \mu(T(u)),$$

where $u \in Cl(\mathcal{H}_{\mathbb{C}}, q_{\mathbb{C}})$, is equivalent to Condition (6.45) [Ara71, Theorem 7]. Any unitary map $U$ on $\mathcal{H}_{\mathbb{C}}^{1,0}$ is extended to $U + \bar{U}$ on $\mathcal{H}_{\mathbb{C}}$. The extension has an invariant subspace $(\mathcal{H}, q)$ and is unitary implementable. In a word, the space of unitary

[1]Note that in this section we use the skewed adjoint representation $\widetilde{Ad}$, then the image of $\overline{Pin}(\mathcal{H}, q)$ does not contain the case $-id + T$. If we use the usual adjoint representation, $-id + T$ will be included

implementable Bogoliubov transformations contains the unitary group $U(\mathcal{H}_{\mathbb{C}}^{1,0})$. Let $T$ be a real unitary map in $U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$. Loosely speaking, it is in the form

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$$

and the form is based on the splitting $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{1,0} \oplus \mathcal{H}_{\mathbb{C}}^{0,1}$. After we modulo $U(\mathcal{H}_{\mathbb{C}})$, $P = |P|$ is self-adjoint and positive. In this case, $T$ is unitary implementable if and only if $T - id$ is Hilbert-Schmidt and the construction in Proposition 6.14 yields an approximation to $T$.

Quasi-free states $\omega$ are related to their generalized one-particle density matrices, which are defined linear operators $S_\omega$ on $(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$ such that

$$\langle v, S_\omega u\rangle_{\mathcal{H}_{\mathbb{C}}} = \mathrm{Tr}_{\mathcal{F}_a}\left(\mu(u)\mu^*(v)\omega\right), \tag{6.46}$$

where $u, v \in (\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$. There is an implementable Bogoliubov transform $T$ such that $\omega = \pi(T)\Omega$ if and only if the generalized one-particle density matrix satisfies [Sol14, Theorem 10.4]

$$S_\omega^* = S_\omega, \quad S_\omega^2 = S_\omega, \quad P_{\mathcal{H}_{\mathbb{C}}^{1,0}} S_\omega P_{\mathcal{H}_{\mathbb{C}}^{1,0}} \in \mathcal{L}^1.$$

where $P_{\mathcal{H}_{\mathbb{C}}^{1,0}}$ is the projection on $\mathcal{H}_{\mathbb{C}}^{1,0}$.

Let explore the relation in special cases. If $\omega = \mu(T^{-1})\,|0\rangle$ for $T \in Pin(\mathcal{H}, q)$, then

$$
\begin{aligned}
&\langle v, S_{\mu(T^{-1})|0\rangle} u\rangle_{\mathcal{H}_{\mathbb{C}}} \\
&= \left\langle |0\rangle, \mu\left(\widetilde{Ad}_T(u)\right)\mu\left(\widetilde{Ad}_T(\bar{v})\right)|0\rangle\right\rangle_{\mathcal{F}_a} \\
&= \left\langle \widetilde{Ad}_T(v), S_{id}\widetilde{Ad}_T(u)\right\rangle_{\mathcal{H}_{\mathbb{C}}}
\end{aligned}
$$

and $S_{\mu(T^{-1})|0\rangle} = \widetilde{Ad}_T^* S_{id}\widetilde{Ad}_T$, where $S_{id}$ is the projection on $\mathcal{H}_{\mathbb{C}}^{0,1}$. Note that $\bar{S}_{id}$ is the projection onto $\mathcal{H}_{\mathbb{C}}^{1,0}$. Therefore $S_{\mu(T^{-1})|0\rangle}$ satisfies identities

$$
S^*_{\mu(T^{-1})|0\rangle} = S_{\mu(T^{-1})|0\rangle}, \quad S^2_{\mu(T^{-1})|0\rangle} = S_{\mu(T^{-1})|0\rangle} \quad \text{and} \quad S_{\mu(T^{-1})|0\rangle} + \bar{S}_{\mu(T^{-1})|0\rangle} = id.
$$

Conversely, we have

**Lemma 6.15.** *Let $L$ be a bounded operator on $\mathcal{H}_{\mathbb{C}}$ and it satisfies*

$$
L^* = L, \quad L^2 = L \quad \text{and} \quad L + \bar{L} = id, \tag{6.47}
$$

*then there is a real unitary operator $U \in U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$ such that $L = U^* S_{id} U$.*

*Proof.* Consider $L - id/2$,

$$
\left(L - \frac{id}{2}\right)^2 = \frac{id}{4},
$$

and the polar decomposition of $L - id/2$,

$$
L - \frac{id}{2} = U_L \frac{id}{2}
$$

where $U_L \in U(\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}})$. For any $u \in \mathcal{H}_{\mathbb{C}}$, we have the decomposition

$$u = \frac{u + U_L u}{2} + \frac{u - U_L u}{2}.$$

By direction computation,

$$\left(L - \frac{id}{2}\right)(u + U_L u) = \frac{1}{2}(u + U_L u),$$
$$\left(L - \frac{id}{2}\right)(u - U_L u) = -\frac{1}{2}(u - U_L u).$$

It means that $\mathcal{H}_{\mathbb{C}}$ can be decomposed as a sum of two eigenspaces $V_{1/2}$ and $V_{-1/2}$ of $L - id/2$. Since $L + \bar{L} = id$, the two eigenspaces are related by the complex conjugation, i.e. $\bar{V}_{1/2} = V_{-1/2}$. This decomposition is similar to $\mathcal{H}_{\mathbb{C}}^{1,0}$ and $\mathcal{H}_{\mathbb{C}}^{0,1}$.

Next we extend a unitary map $U$ from

$$\mathcal{H}_{\mathbb{C}} = V_{-1/2} \oplus V_{1/2} \quad \text{to} \quad \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{1,0} \oplus \mathcal{H}_{\mathbb{C}}^{0,1}$$

as follows: construct a unitary map $U_1$ from $V_{-1/2}$ to $\mathcal{H}_{\mathbb{C}}^{1,0}$, then extend it to $V_{1/2}$ using the complex conjugation, i.e.

$$U_2 u := \overline{U_1 \bar{u}}, \quad u \in V_{1/2}.$$

$U$ is the sum of $U_1$ and $U_2$. Due to the construction of $U$, $U$ is real and

$$L - \frac{id}{2} = U^* \left( -\frac{id|_{\mathcal{H}_{\mathbb{C}}^{1,0}}}{2} + \frac{id|_{\mathcal{H}_{\mathbb{C}}^{0,1}}}{2} \right) U.$$

□

Slater determinants are states in $\bigwedge^n(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ in the form

$$u_1 \wedge u_2 \wedge \ldots \wedge u_n,$$

where $u_j$ are orthonormal $\|u_j\|_{\mathcal{H}} = q(u_j) = 1$. In physics, they are state functions of $n$ Fermions.

**Corollary 6.16.** *Slater determinants are pure quasi-free states.*

*Proof.* Consider the state $u_1 \wedge u_2 \wedge \ldots \wedge u_n$, where $u_j$ are orthonormal $\|u_j\|_{\mathcal{H}} = q(u_j) = 1$. Then $u_1 \cdots u_n \in Pin(\mathcal{H}, q)$. Recall the decomposition of $u_j$ in $\mathcal{H}_{\mathbb{C}}$,

$$u_j = \frac{u_j - iJu_j}{2} + \frac{u_j + iJu_j}{2},$$

and $\mu(u_j) = a^\dagger(u_j) + a(u_j)$. Therefore $u_1 \wedge u_2 \wedge \ldots \wedge u_n = \mu(u_1)\mu(u_2)\ldots\mu(u_n)\,|0\rangle$ and it is a pure quasi-free state. □

We show an approximation result on the skewed adjoint representation, which is used in Section 6.2.3.

**Lemma 6.17.** *Consider* $\widetilde{Ad}_{(u_1 u_2 \ldots u_n)}$ *and* $\widetilde{Ad}_{(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n)}$ *with* $u_j, \tilde{u}_j \in (\mathcal{H}, q)$ *and* $\|u_j\|_{\mathcal{H}} = \|\tilde{u}_j\|_{\mathcal{H}} = 1$ *for* $1 \le j \le n$*, then*

$$\left\|\widetilde{Ad}_{(u_1 u_2 \ldots u_n)} - \widetilde{Ad}_{(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n)}\right\|_{\mathcal{L}^1} \le 4 \sum_{j=1}^{n} \|u_j - \tilde{u}_j\|_{\mathcal{H}}.$$

*Proof.* Consider the basic case $n = 1$,

$$\left(\widetilde{Ad}_{(u)} - \widetilde{Ad}_{(\tilde{u})}\right)(v) = -2q_{\mathbb{C}}(v,u)u + 2q_{\mathbb{C}}(v,\tilde{u})\tilde{u}, \quad v \in (\mathcal{H}_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathbb{C}}}),$$

and

$$\begin{aligned}
&\left\|\widetilde{Ad}_{(u)} - \widetilde{Ad}_{(\tilde{u})}\right\|_{\mathcal{L}^1} \\
\leq &\left\|(h \mapsto 2q_{\mathbb{C}}(h,u)(u-\tilde{u}))\right\|_{tr} + \left\|(h \mapsto 2q_{\mathbb{C}}(h,u-\tilde{u}))\tilde{u}\right\|_{tr} \\
= &4\|u - \tilde{u}\|
\end{aligned}$$

To prove the general case, use the observation for $h_1, h_2 \in Cl^{\times}(H_{\mathbb{C}}, q_{\mathbb{C}})$,

$$\begin{aligned}
&\left\|\widetilde{Ad}_{(h_1 u h_2)} - \widetilde{Ad}_{(h_1 \tilde{u} h_2)}\right\|_{\mathcal{L}^1} \\
= &\left\|\widetilde{Ad}_{(h_1)}\left(\widetilde{Ad}_{(u)} - \widetilde{Ad}_{(\tilde{u})}\right)\widetilde{Ad}_{(h_2)}\right\|_{\mathcal{L}^1} \\
\leq &\left\|\widetilde{Ad}_{(h_1)}\right\|_{op}\left\|\widetilde{Ad}_{(u)} - \widetilde{Ad}_{(\tilde{u})}\right\|_{\mathcal{L}^1}\left\|\widetilde{Ad}_{(h_2)}\right\|_{op}.
\end{aligned}$$

□

### 6.2.3 Infinite Dimensional Case

In this section, we consider a special case when $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C})$ and study smooth approximation of pure quasi-free states.

The real structure of $\mathcal{H}$ can be viewed as $L^2(\mathbb{R}^d, \mathbb{R}^2)$, i.e. elements are written in the form $\begin{pmatrix} f \\ g \end{pmatrix}$, $f, g \in L^2(\mathbb{R}^d, \mathbb{R})$. Then the quadratic form $q$ is

$$q\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) := \int_{\mathbb{R}^d} dx \left(f_1(x) f_2(x) + g_1(x) g_2(x)\right),$$

where $f_j, g_j \in L^2(\mathbb{R}^n, \mathbb{R}), j = 1, 2$, and the complex structure $J$ is

$$J \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -g \\ f \end{pmatrix},$$

where $f, g \in L^2(\mathbb{R}^d, \mathbb{R})$. The Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} dx\, \bar{f}(x) g(x)$$

where $f, g \in L^2((\mathbb{R}^d, \mathbb{C}))$. $(\mathcal{H}_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}})$ is identified with $\mathcal{H} \times \mathcal{H}$ in the way: $\mathcal{H}_{\mathbb{C}}^{1,0}$ is identified with $(\mathcal{H}, J)$, i.e. $\mathcal{H}$ with complex structure $J$,

$$\frac{u - iJu}{2} \mapsto u \in (\mathcal{H}, J) \sim f + ig$$

and $\mathcal{H}_{\mathbb{C}}^{0,1}$ is identified with $(\mathcal{H}, -J)$, i.e. $\mathcal{H}$ with complex structure $-J$

$$\frac{u + iJu}{2} \mapsto u \in (\mathcal{H}, -J) \sim f - ig,$$

where $u = \begin{pmatrix} f \\ g \end{pmatrix}$, $f, g \in L^2(\mathbb{R}^d, \mathbb{R})$. This identification is complex linear. Under the identification the complex conjugation on $\mathcal{H}_{\mathbb{C}}$ corresponds to $\mathcal{J}$ on $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$,

$$\mathcal{J} \begin{pmatrix} F \\ G \end{pmatrix} := \begin{pmatrix} \bar{G} \\ \bar{F} \end{pmatrix}, \tag{6.48}$$

where $F, G \in L^2(\mathbb{R}^d, \mathbb{C})$. $q_{\mathbb{C}}$ on $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$ assumes the form

$$q_{\mathbb{C}} \left( \begin{pmatrix} F_1 \\ G_1 \end{pmatrix}, \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} \right) = \frac{1}{2} \int_{\mathbb{R}^d} dx\, F_1(x) G_2(x) + G_1(x) F_2(x), \tag{6.49}$$

where $F_j, G_j \in L^2(\mathbb{R}^d, \mathbb{C}), j = 1, 2$, and the Hermitian form on $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$ is

$$\left\langle \begin{pmatrix} F_1 \\ G_1 \end{pmatrix}, \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} \right\rangle_{\mathcal{H} \times \mathcal{H}} = \int_{\mathbb{R}^d} dx \bar{F}_1(x) F_2(x) + \bar{G}_1(x) G_2(x). \tag{6.50}$$

The action $\mu$ is

$$\mu \left( \begin{pmatrix} F \\ G \end{pmatrix} \right) = \int_{\mathbb{R}^d} dx\, \left( F(x) a_x^\dagger + G(x) a_x \right). \tag{6.51}$$

$(\mathcal{H}, q)$ corresponds to elements in the form $\begin{pmatrix} F \\ \bar{F} \end{pmatrix}$, $F \in L^2(\mathbb{R}^d, \mathbb{C})$, which form an invariant subspace of $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$ under $\mathcal{J}$. Besides $q\left( \begin{pmatrix} F \\ \bar{F} \end{pmatrix} \right) = 1$ is then equivalent to $\|F\|_{L^2(\mathbb{R}^d)} = 1$.

**Example 6.18.** We compute integral kernels explicitly in two basic cases

1. $\widetilde{Ad}_{(u)}$, where $u = \begin{pmatrix} F \\ \bar{F} \end{pmatrix}$, $F \in L^2(\mathbb{R}^d, \mathbb{C})$ and $\|F\|_{L^2(\mathbb{R}^d)} = 1$ .

2. $d\widetilde{Ad}_{(u_1 u_2 - q_{\mathbb{C}}(u_1,u_2))}$, where $u_j = \begin{pmatrix} F_j \\ \bar{F}_j \end{pmatrix}$, $F_j \in L^2(\mathbb{R}^d, \mathbb{C})$ and $\|F_j\|_{L^2(\mathbb{R}^d)} = 1$, $j = 1, 2$.

Let $h = \begin{pmatrix} f \\ g \end{pmatrix} \in L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$. For case 1.

$$\begin{aligned}
&\widetilde{Ad}_{(u)}(h) \\
=&h - 2q_{\mathbb{C}}(u, h)u \\
=&\begin{pmatrix} f \\ g \end{pmatrix} - \left( \int_{\mathbb{R}^n} dy\, f(y)\bar{F}(y) + F(y)g(y) \right) \begin{pmatrix} F \\ \bar{F} \end{pmatrix},
\end{aligned}$$

then its integral kernel $K_{\widetilde{Ad}_{(u)}}$ is

$$K_{\widetilde{Ad}_{(u)}}(x,y) = \begin{pmatrix} \delta(x-y) & 0 \\ 0 & \delta(x-y) \end{pmatrix} - \begin{pmatrix} F(x)\bar{F}(y) & F(x)F(y) \\ \bar{F}(x)\bar{F}(y) & \bar{F}(x)F(y) \end{pmatrix}.$$

For case 2.

$$\begin{aligned}
&d\widetilde{Ad}_{(u_1 u_2 - q_{\mathbb{C}}(u_1,u_2))}(u) \\
=&2q_{\mathbb{C}}(u_2, u)u_1 - 2q_{\mathbb{C}}(u_1, u)u_2 \\
=&\left( \int_{\mathbb{R}^n} dy\, f(y)\bar{F}_2(y) + g(y)F_2(y) \right) \begin{pmatrix} F_1 \\ \bar{F}_1 \end{pmatrix} - \left( \int_{\mathbb{R}^n} dy\, f(y)\bar{F}_1(y) + g(y)F_1(y) \right) \begin{pmatrix} F_2 \\ \bar{F}_2 \end{pmatrix},
\end{aligned}$$

then its integral kernel $K_{d\widetilde{Ad}_{(u_1u_2-q_{\mathbb{C}}(u_1,u_2))}}$ is

$$K_{d\widetilde{Ad}_{(u_1u_2-q_{\mathbb{C}}(u_1,u_2))}}(x,y)=\begin{pmatrix} F_1(x)\bar{F}_2(y)-F_2(x)\bar{F}_1(y) & F_1(x)F_2(y)-F_2(x)F_1(y) \\ \bar{F}_1(x)\bar{F}_2(y)-\bar{F}_2(x)\bar{F}_1(y) & \bar{F}_1(x)F_2(y)-\bar{F}_2(x)F_1(y) \end{pmatrix}.$$

Meanwhile the infinitesimal representation $d\mu$

$$\begin{aligned}
&\mu\left(u_1u_2-q_{\mathbb{C}}(u_1,u_2)\right) \\
=&\int dxdy \begin{pmatrix} a_x^\dagger & a_x \end{pmatrix}\begin{pmatrix} F_1(x)\bar{F}_2(y) & F_1(x)F_2(y) \\ \bar{F}_1(x)\bar{F}_2(y) & \bar{F}_1(x)F_2(y) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_y^\dagger \\ a_y \end{pmatrix} \\
&-\frac{1}{2}\int_{\mathbb{R}^n} dy\, F_1(y)\bar{F}_2(y)+\bar{F}_1(y)F_2(y) \\
=&\frac{1}{2}\int dxdy \begin{pmatrix} a_x^\dagger & a_x \end{pmatrix} K_{d\widetilde{Ad}_{(u_1u_2-q_{\mathbb{C}}(u_1,u_2))}}(x,y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_y^\dagger \\ a_y \end{pmatrix}.
\end{aligned}$$

Let $T$ be an operator on $L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C})$. If $T$ commutes with the action $\mathcal{J}$, i.e. $T\mathcal{J}=\mathcal{J}T$, it is in the form

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P,Q\in\mathcal{B}\left(L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C})\right),$$

where $\bar{P}u := \overline{P\bar{u}}$, $u\in L^2(\mathbb{R}^d,\mathbb{C})$. If $P$ has an integral kernel, $\bar{P}$ means taking the complex conjugation of its integral kernel. Formally, the infinitesimal representation

$d\mu$

$$d\mu : \mathfrak{spin}(\mathcal{H}, q) \to \mathfrak{u}(\mathcal{F}_a)$$

$$\begin{pmatrix} P(x,y) & Q(x,y) \\ \bar{Q}(x,y) & \bar{P}(x,y) \end{pmatrix} \mapsto \frac{1}{2}\int dxdy \begin{pmatrix} a_x^\dagger & a_x \end{pmatrix} \begin{pmatrix} P(x,y) & Q(x,y) \\ \bar{Q}(x,y) & \bar{P}(x,y) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_y^\dagger \\ a_y \end{pmatrix}$$

where $\bar{P}(y,x) = -P(x,y)$ and $Q(y,x) = -Q(x,y)$.

In the end, we study the smooth approximation of pure quasi-free states.

**Lemma 6.19.** *Consider* $\widetilde{Ad}_{(u_1 u_2 \ldots u_n)}$ *and* $\widetilde{Ad}_{(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n)}$ *with* $u_j = \begin{pmatrix} f_j \\ \bar{f}_j \end{pmatrix}, \tilde{u}_j = \begin{pmatrix} \tilde{f}_j \\ \bar{\tilde{f}}_j \end{pmatrix}$ *and* $\|f_j\|_{L^2(\mathbb{R}^d)} = \|\tilde{f}_j\|_{L^2(\mathbb{R}^d)} = 1$ *for* $1 \le j \le n$*, then*

$$\left\| \left( \widetilde{Ad}_{(u_1 u_2 \ldots u_n)} - \widetilde{Ad}_{(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1(L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C}))} \le C \sum_{j=1}^{n} \left\| f_j - \tilde{f}_j \right\|_{H^1(\mathbb{R}^d)},$$

*where $C$ is a constant depending on $n$, $\|f_j\|_{H^1(\mathbb{R}^d)}$ and the differential operator $\langle \nabla \rangle$ acts on $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$ diagonally.*

*Proof.* Consider the simplest case, $n = 1$,

$$\begin{aligned} &\left\| \left( \widetilde{Ad}_{(u_1)} - \widetilde{Ad}_{(\tilde{u}_1)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\ \le &\left\| (h \mapsto 2q_{\mathbb{C}}(u_1 - \tilde{u}_1, h)\tilde{u}_1) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\ &+ \left\| (h \mapsto 2q_{\mathbb{C}}(u_1, h)(u_1 - \tilde{u}_1)) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\ = &2 \left\| f_1 - \tilde{f}_1 \right\|_{H^1(\mathbb{R}^d)} + 2\|f_1\|_{H^1(\mathbb{R}^d)} \left\| f_1 - \tilde{f}_1 \right\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

In order to show the general case, note that for $h_1, h_2 \in Pin(\mathcal{H}, q)$,

$$
\begin{aligned}
&\left\| \left( \widetilde{Ad}_{(h_1 u_1 h_2)} - \widetilde{Ad}_{(h_1 \tilde{u}_1 h_2)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\
\leq &\left\| \widetilde{Ad}_{(h_1)} \right\|_{op} \left\| \left( \widetilde{Ad}_{(u_1)} - \widetilde{Ad}_{(\tilde{u}_1)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\
&+ \left\| \widetilde{Ad}_{(h_1)} \right\|_{op} \left\| \widetilde{Ad}_{(u_1)} - \widetilde{Ad}_{(\tilde{u}_1)} \right\|_{\mathcal{L}^1} \left\| \left( \widetilde{Ad}_{(h_2)} - id \right) \langle \nabla \rangle \right\|_{op} \\
\leq &\left\| \widetilde{Ad}_{(h_1)} \right\|_{op} \left( 2 \left\| f_1 - \tilde{f}_1 \right\|_{H^1(\mathbb{R}^d)} + 2 \| f_1 \|_{H^1} \left\| f_1 - \tilde{f}_1 \right\|_{L^2(\mathbb{R}^d)} \right) \\
&+ 4 \left\| \widetilde{Ad}_{(h_1)} \right\|_{op} \left\| f_1 - \tilde{f}_1 \right\|_{L^2(\mathbb{R}^d)} \left\| \left( \widetilde{Ad}_{(h_2)} - id \right) \langle \nabla \rangle \right\|_{op} \quad \text{(Lemma 6.17)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left\| \left( \widetilde{Ad}_{(u_1 u_2 \ldots u_n)} - \widetilde{Ad}_{(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\
\leq &\sum_{j=1}^{n} \left\| \left( \widetilde{Ad}_{(\tilde{u}_1 \ldots \tilde{u}_{j-1} u_j \ldots u_n)} - \widetilde{Ad}_{(\tilde{u}_1 \ldots \tilde{u}_j u_{j+1} \ldots u_n)} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^1} \\
\leq &C \sum_{j=1}^{n} \left\| f_j - \tilde{f}_j \right\|_{H^1(\mathbb{R}^d)},
\end{aligned}
$$

where $C$ is a constant depending on $n$ and $\|f_j\|_{H^1(\mathbb{R}^d)}$. □

**Proposition 6.20.** *Let $T$ be a unitary implementable Bogoliubov transform on $L^2(\mathbb{R}^d, \mathbb{C}) \times L^2(\mathbb{R}^d, \mathbb{C})$ such that $T - id$ is a Hilbert-Schmidt operator, there is a sequence of smooth operators $\tilde{T}_n \in Pin(\mathcal{H}, q)$, such that the integral kernels of*

$$
\widetilde{Ad}_{\tilde{T}_n} - \begin{pmatrix} id_{L^2(\mathbb{R}^d, \mathbb{C})} & 0 \\ 0 & id_{L^2(\mathbb{R}^d, \mathbb{C})} \end{pmatrix}
$$

*are smooth compactly supported functions and*

$$\left\|\left(T - \widetilde{Ad}_{\tilde{T}_n}\right)\langle\nabla\rangle\right\|_{\mathcal{L}^2(L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C}))} \to 0.$$

*where the differential operator $\langle\nabla\rangle$ acts on $L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C})$ diagonally.*

*Proof.* Apply the finite-rank approximation in Proposition 6.14 to $\widetilde{Ad}_T$ and denote the approximation operator by $\widetilde{Ad}_{T_n}$, where $T_n = u_1u_2\dots u_m$ and $u_j \in (\mathcal{H}, q)$. Use the definition of the Hilbert-Schmidt norm,

$$\begin{aligned}
&\left\|\left(T - \widetilde{Ad}_{T_n}\right)\langle\nabla\rangle\right\|^2_{\mathcal{L}^2(L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C}))}\\
=&\mathrm{Tr}\left(\left(T - \widetilde{Ad}_{T_n}\right)\langle\nabla\rangle^2\left(T - \widetilde{Ad}_{T_n}\right)^*\right)\\
=&\mathrm{Tr}_{V_{n\mathbb{C}}}\left(\left(T - \widetilde{Ad}_{T_n}\right)\langle\nabla\rangle^2\left(T - \widetilde{Ad}_{T_n}\right)^*\right)\\
&+\mathrm{Tr}_{V_{n\mathbb{C}}^{\perp}}\left(\left(T - \widetilde{Ad}_{T_n}\right)\langle\nabla\rangle^2\left(T - \widetilde{Ad}_{T_n}\right)^*\right)\\
=&\mathrm{Tr}_{V_{n\mathbb{C}}^{\perp}}\left((T-id)\langle\nabla\rangle^2(T-id)^*\right),
\end{aligned}$$

where $\mathrm{Tr}_{V_{n\mathbb{C}}^{\perp}}\left((T-id)\langle\nabla\rangle^2(T-id)^*\right)$ approaches zero, as $n\to\infty$.

Next we modify the finite-rank approximation $\widetilde{Ad}_{T_n}$. Approximate $u_j$ by compactly supported smooth functions $\tilde{u}_j \in (\mathcal{H}, q)$ with respect to $H^1(\mathbb{R}^d,\mathbb{C})$. The resulting operator

$$\widetilde{Ad}_{\tilde{T}_n} - id_{L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C})}$$

has a compactly supported smooth kernel, where $\tilde{T}_n = \tilde{u}_1\tilde{u}_2\dots\tilde{u}_m$, and by Lemma

6.19,

$$\left\| \left( \widetilde{Ad}_{\tilde{T}_n} - \widetilde{Ad}_{T_n} \right) \langle \nabla \rangle \right\|_{\mathcal{L}^2(L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C}))} \to 0.$$

□

Let $\omega = |\mu(T^{-1})|0\rangle\rangle \langle \mu(T^{-1})|0\rangle|$ be a quasi-free state and its generalized one-particle density matrix is

$$S_\omega = \begin{pmatrix} \Gamma & \Lambda \\ \Lambda^* & 1 - \bar{\Gamma} \end{pmatrix}$$

Now we fix the generalized one-particle density matrix. Without loss of generality,

$$\widetilde{Ad}_T = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix},$$

where $P$ is semi-positive, i.e. $P = |P|$. If $P$ is not equal to $|P|$, consider its polar decomposition $P = U|P|$

$$\left( \begin{pmatrix} U^{-1} & 0 \\ 0 & \bar{U}^{-1} \end{pmatrix} \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \right)^* S_{id} \begin{pmatrix} U^{-1} & 0 \\ 0 & \bar{U}^{-1} \end{pmatrix} \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$$
$$= \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}^* S_{id} \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$$

i.e. $\begin{pmatrix} |P| & U^{-1}Q \\ \bar{U}^{-1}\bar{Q} & |\bar{P}| \end{pmatrix}$ has the same generalized one-particle density matrix and it responds to the same quasi-free state up to a phase. Recall $S_\omega = \widetilde{Ad}_T^* S_{id} \widetilde{Ad}_T$,

$\Gamma = \bar{Q}^*\bar{Q}$ and $\Lambda = \bar{Q}^*\bar{P}$, where $S_{id}$ is the identity on $\mathcal{H}^{0,1}_{\mathbb{C}}$. Suppose the generalized one-particle density matrix satisfies

$$\Gamma \in \mathcal{L}^{1,1}\left(L^2(\mathbb{R}^d,\mathbb{C})\right) \quad \text{and} \quad \Lambda \in H^1\left(\mathbb{R}^d \times \mathbb{R}^d,\mathbb{C}\right).$$

The conditions pass to $Q$ and $P$

$$\langle\nabla\rangle\bar{Q}^*\bar{Q}\langle\nabla\rangle \in \mathcal{L}^1\left(L^2(\mathbb{R}^d,\mathbb{C}\right) \quad \text{and} \quad \bar{Q}^*\bar{P} \in H^1\left(\mathbb{R}^d \times \mathbb{R}^d,\mathbb{C}\right).$$

Then $Q\langle\nabla\rangle \in \mathcal{L}^2\left(L^2(\mathbb{R}^3,\mathbb{C})\right)$ and $(P - id)\langle\nabla\rangle \in \mathcal{L}^1\left(L^2(\mathbb{R}^3,\mathbb{C})\right)$, since

$$id - P^*P = \bar{Q}^*\bar{Q} \implies (P - id)\langle\nabla\rangle = -(P + id)^{-1}\bar{Q}^*\bar{Q}\langle\nabla\rangle.$$

Therefore

$$\left(\widetilde{Ad}_T - id_{L^2(\mathbb{R}^d,\mathbb{C})\times L^2(\mathbb{R}^d,\mathbb{C})}\right)\langle\nabla\rangle = \begin{pmatrix} P - id & Q \\ \bar{Q} & \bar{P} - id \end{pmatrix}\langle\nabla\rangle \in \mathcal{L}^2(L^2(\mathbb{R}^d,\mathbb{C}) \times L^2(\mathbb{R}^d,\mathbb{C})).$$

Use Proposition 6.20, $\widetilde{Ad}_{\tilde{T}_n}$ converges to $\widetilde{Ad}_T$ and $S_{\mu(\tilde{T}_n^{-1})|0\rangle}$ converges to $S_\omega$ in the sense that, the convergence of the first entry is in the trace norm and the convergence of the second entry is in the Hilbert-Schmidt norm.

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